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Integrability of Gaussian Multiplicative Chaos and Liouville Conformal Field Theory

Intégrabilité du Chaos Multiplicatif Gaussien et Théorie Conforme des Champs de Liouville

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Intégrabilité du Chaos Multiplicatif Gaussien et Théorie Conforme des Champs de Liouville

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Résumé

Cette thèse de doctorat porte sur l'étude de deux objets probabilistes, les mesures de chaos multiplicatif gaussien (GMC) et la théorie conforme des champs de Liouville (LCFT). Le GMC fut introduit par Kahane en 1985 et il s'agit aujourd'hui d'un objet extrêmement important en théorie des probabilités et en physique mathématique. Très récemment le GMC a été utilisé pour définir les fonctions de corrélation de la LCFT, une théorie qui est apparue pour la première fois en 1981 dans le célèbre article de Polyakov, "Quantum geometry of bosonic strings".

Grâce à ce lien établi entre GMC et LCFT, nous pouvons traduire les techniques de la théorie conforme des champs dans un langage probabiliste pour effectuer des calculs exacts sur les mesures de GMC. Ceci est précisément ce que nous développerons pour le GMC sur le cercle unité. Nous écrirons les équations BPZ qui fournissent des relations non triviales sur le GMC. Le résultat final est la densité de probabilité pour la masse totale de la mesure de GMC sur le cercle unité ce qui résout une conjecture établie par Fyodorov et Bouchaud en 2008. Par ailleurs, il s'avère que des techniques similaires permettent également de traiter un autre cas, celui du GMC sur le segment unité, et nous obtiendrons de même des formules qui avaient été conjecturées indépendamment par Ostrovsky et par Fyodorov, Le Doussal, et Rosso en 2009.

La dernière partie de cette thèse consiste en la construction de la LCFT sur un domaine possédant la topologie d'une couronne. Nous suivrons les méthodes introduites par David-Kupiainen-Rhodes-Vargas même si de nouvelles techniques seront requises car la couronne possède deux bords et un espace des modules non trivial. Nous donnerons également des preuves plus concises de certains résultats connus.

Abstract

Throughout this PhD thesis we will study two probabilistic objects, Gaussian multiplicative chaos (GMC) measures and Liouville conformal field theory (LCFT). GMC measures were first introduced by Kahane in 1985 and have grown into an extremely important field of probability theory and mathematical physics. Very recently GMC has been used to give a probabilistic definition of the correlation functions of LCFT, a theory that first appeared in Polyakov's 1981 seminal work, "Quantum geometry of bosonic strings".

Once the connection between GMC and LCFT is established, one can hope to translate the techniques of conformal field theory in a probabilistic framework to perform exact computations on the GMC measures. This is precisely what we develop for GMC on the unit circle. We write down the BPZ equations which lead to non-trivial relations on the GMC. Our final result is an exact probability density for the total mass of the GMC measure on the unit circle. This proves a conjecture of Fyodorov and Bouchaud stated in 2008. Furthermore, it turns out that the same techniques also work on a more difficult model, the GMC on the unit interval, and thus we also prove conjectures put forward independently by Ostrovsky and by Fyodorov, Le Doussal, and Rosso in 2009.

The last part of this thesis deals with the construction of LCFT on a domain with the topology of an annulus. We follow the techniques introduced by David-Kupiainen-Rhodes-Vargas although novel ingredients are required as the annulus possesses two boundaries and a non-trivial moduli space. We also provide more direct proofs of known results.

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Contents

Chapter 1

An overview of Gaussian chaos and Liouville theory

1.1 Introduction

The present chapter is destined to give an overview of the results concerning two probabilistic objects, the Gaussian multiplicative chaos (GMC) measures and Liouville conformal field theory (LCFT). We will present the definitions, the most important properties and the required information the reader needs to understand the following chapters. The theory of Gaussian multiplicative chaos was first introduced heuristically by B. Mandelbrot in 1972 in [50, 51] and then defined rigorously by J.-P. Kahane in 1985 in [42] and has ever since been extensively studied in many problems of probability theory and mathematical physics with applications including random geometry, 2d quantum gravity, statistical physics, 3d turbulence and mathematical finance. See [72] for a review.

On the other hand Liouville conformal field theory was only very recently understood as a probabilistic object. It first appeared in the physics literature in the seminal paper "Quantum geometry of bosonic strings" by A. Polyakov in 1981 [66]. Considerable efforts were then devoted by theoretical physicists to try to solve Liouville theory - in other words to compute its correlation functions - motivated by its importance in the study of non-critical string theory and of random geometry in two dimensions. A major step in this direction was taken by Belavin, Polyakov, and Zamolodchikov (BPZ) in 1984 in the paper [8] which laid down the foundations of conformal field theory (CFT).[1] Liouville field theory is indeed a CFT and using these techniques the celebrated DOZZ formula was conjectured independently by Dorn and Otto in [22] and Zamolodchikov and Zamolodchikov in [86]. This formula gives the value of the most fundamental quantity in Liouville theory: the three-point correlation function of the theory on the Riemann sphere. However, the major problem behind this study of Liouville theory was that it completely lacked mathematical rigour and the derivation of the DOZZ formula was seen more as a guess even on a physics level of rigour. The biggest problem came from the fact that it appeared unclear how to construct a consistent theory based on the so-called path integral formalism in the case of Liouville theory. Many reviews on LCFT abandon this path integral definition right from the start and replace it by purely algebraic definitions. This is the famous conformal bootstrap approach to LCFT, see [71].

[1] The original motivation of BPZ in [8] was precisely to obtain an exact value for the correlation function of LCFT although CFT has now grown into a huge field of theoretical physics with countless applications.

A decisive step to solve this problem was made by David-Kupiainen-Rhodes-Vargas in the paper [17] where a probabilistic definition is given for the Liouville field theory. The authors find a way to interpret the path integral of LCFT: using a regularization and renormalization procedure they obtain a rigorous probabilistic definition of the theory. The correlation functions of LCFT are thus expressed in terms of moments of GMC measures with log-singularities. We emphasize that this establishes a link that was unknown to two major communities of physicists, the statistical physics community working with GMC related models and the theoretical physics community working on LCFT.

Once a probabilistic content has been given to the correlation functions of LCFT, the next natural step is to try to prove all the properties one expects for a CFT. The correlation functions of LCFT are shown to behave as conformal tensors (KPZ formula) and obey the Weyl anomaly (behaviour under a change of background metric). Next we can introduce the stress-energy tensor of the theory and prove the Ward identities. Lastly the BPZ equations are established for a correlation function where at least one of the point in the correlation has a so-called degenerate weight. These differential equations are extremely important as they translate the constraints imposed by the local conformal invariance (by opposition to the KPZ formula which just takes into account global conformal invariance).

Using these BPZ equations CFT tells us that it should be possible to perform exact computations of certain correlation functions with a small number of points. In the case of Liouville theory the precise way to extract information out of the BPZ equations to obtain non-trivial relations on the correlation functions goes back to Teschner [80]. On the Riemann sphere the simplest correlation is the three-point function and the BPZ equations thus lead to relations that completely determine this function, this is the content of the celebrated DOZZ formula. Implementing all of the above in a probabilistic framework allowed Kupiainen-Rhodes-Vargas in 2017 to give a proof of the DOZZ formula [46, 47]. In chapter 2 we will explain how writing down the BPZ equations for a domain with boundary allows one to obtain the law of the total mass of the GMC measure on the unit circle, the so-called Fyodorov-Bouchaud formula [31].

To say a little bit more on the Fyodorov-Bouchaud formula let us take a small detour in the world of statistical physics. In this world GMC measures were first considered on the interval in the work of Bacry-Muzy [7] followed by the work of Fyodorov-Bouchaud [31] on a random energy model, which they studied on the circle. Fyodorov and Bouchaud conjectured the law of the total mass on the circle using a heuristic analytic continuation from integer to complex moments which we refer to as the Fyodorov-Bouchaud formula. They also conjectured the distribution of the maximum of underlying gaussian field. The Fyodorov-Bouchaud formula was extended by Ostrovsky [62] to include a single insertion point. The case of the distribution of the total mass of the GMC measure on the interval was first considered by Ostrovsky [58] using his theory of intermittency differentiation resulting in explicit conjectures for the negative moments [59] and for the fractional moments of the total mass [60]. The problem of the total mass on the interval with two insertion points was independently considered by Fyodorov et al. [35] using the technique of analytic continuation from integer to complex moments. Fyodorov et al. conjectured the fractional moments of the total mass as well as the Laplace transform of the distribution of the maximum of the underlying log-correlated field, see also [34]. The two approaches to the problem with two insertion points were unified in [61], see also [63] for a detailed review. We will prove a number of these conjectures by using the connection with LCFT which allows us to introduce the correct auxiliary functions corresponding to holomorphic observables of CFT.

We now briefly summarize the main results of this thesis.

⋄ **Main result 1:** *Content of chapter 2.*
Consider the log-correlated field X on the unit circle $\partial\mathbb{D}$ with covariance:

$$\mathbb{E}[X(e^{i\theta})X(e^{i\theta'})] = 2\ln\frac{1}{|e^{i\theta}-e^{i\theta'}|}. \quad (1.1.1)$$

We will prove the following exact formula for the associated Gaussian multiplicative chaos measure on the unit circle, for $\gamma \in (0,2)$ and $p < \frac{4}{\gamma^2}$,

$$\mathbb{E}[(\frac{1}{2\pi}\int_0^{2\pi} e^{\frac{\gamma}{2}X(e^{i\theta})}d\theta)^p] = \frac{\Gamma(1-\frac{p\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})^p}. \quad (1.1.2)$$

This formula was first conjectured by Fyodorov and Bouchaud in [31]. In the subsection 1.5.2 of this overview we will sketch the proof of this formula which heavily relies on studying Liouville conformal field theory on the unit disk $\mathbb{D}$.

⋄ **Main result 2:** *Content of chapter 3, in collaboration with Tunan Zhu.*
We provide a formula analogue to (1.1.2) but for the GMC measure on the unit interval $[0,1]$. More precisely let X_I be the log-correlated field on $[0,1]$ with covariance:

$$\mathbb{E}[X_I(x)X_I(y)] = 2\ln\frac{1}{|x-y|}. \quad (1.1.3)$$

Then for $\gamma \in (0,2)$ and for real numbers a, b, p satisfying suitable bounds we will give the exact value of the following quantity:

$$M(\gamma,p,a,b) := \mathbb{E}[(\int_0^1 x^a(1-x)^b e^{\frac{\gamma}{2}X_I(x)}dx)^p]. \quad (1.1.4)$$

The strategy of proof resembles strongly the one used to show (1.1.2) although the striking fact is that it is a priori not clear what is the CFT model behind GMC on the interval $[0,1]$. Subsection 1.5.3 summarizes this result.

⋄ **Main result 3:** *Content of chapter 4.*
Our third result differs in spirit from the previous two and deals with the construction of the Liouville field theory on a domain with the topology of an annulus. We first check that the theory verifies the basic properties of a CFT and then integrate the partition function over the moduli space (the space of non conformally equivalent annuli). This allows to give the law of the random modulus and state the conjectured link with random planar maps on such domains. We summarize this construction in more details in subsection 1.3.3.

The remaining part of this overview is divided as follows. First we will give the definitions and basic properties of all the objects that we will use, the Gaussian free field, the Gaussian multiplicative chaos measures and the correlation functions of LCFT. Then we will explain in more details all the properties Liouville field theory inherits from the framework of CFT. Exploiting the constraints imposed by the BPZ equations leads to the derivation of exact formulas for GMC measures and Liouville theory. This is what one should understand by the word "integrability" that we have used in the title, the existence of exact formulas for fractional moments of certain GMC measures or equivalently for the corresponding correlation functions of LCFT. Applications to related problems in probability theory are also included. Lastly we will state some open problems for future directions of research as well as a list of additional exact formulas that we expect to be able to prove using similar techniques.

1.2 Gaussian free fields and Gaussian multiplicative chaos

1.2.1 Gaussian free fields and log-correlated fields

The fundamental Gaussian processes that we will always work with are log-correlated fields which can be defined on $\mathbb{R}^d$ in any dimension d. However for our purpose we will only need the cases $d = 2$ or $d = 1$. In the case of dimension two the field is then known as the Gaussian free field (GFF). In dimension one a log-correlated field can be seen as the restriction to a 1d domain of a Gaussian free field in two-dimensions. For the purpose of this overview and for the coming chapters we will consider many different domains, the Riemann sphere $\mathbb{S}^2$, the unit disk $\mathbb{D}$, the upper-half plane $\mathbb{H}$ and the annulus Ω for the two-dimensional cases and the unit circle $\partial\mathbb{D}$ and the unit interval $[0,1]$ for the one-dimensional cases. Although the general idea is always the same, defining the field on each domain will be slightly different based on the topology of the domain. We will try to treat all the above cases in the most concise way possible.

Let us start with the case of the full plane GFF $\overline{X}$ that will be used to define the GFF on the Riemann phere $\mathbb{S}^2$. One should keep in mind that a very natural way to represent the sphere $\mathbb{S}^2$ is simply to add a point at infinity to the complex plane, i.e. $\mathbb{C} \cup \{\infty\}$. We start by introducing the space $\mathcal{S}(\mathbb{C})$ of smooth test functions on $\mathbb{C}$ with compact support and the subspace $\mathcal{S}_0(\mathbb{C}) \subset \mathcal{S}(\mathbb{C})$ of test functions that have zero average over $\mathbb{C}$. We denote by $\mathcal{S}'(\mathbb{C})$ the space of distributions associated to $\mathcal{S}(\mathbb{C})$ and by $\mathcal{S}'(\mathbb{C})/\mathbb{R}$ the space of distributions associated to $\mathcal{S}_0(\mathbb{C})$ which can be seen as a space of distributions modulo constants.

We can now define the full plane GFF $\overline{X}$ as a Gaussian random variable living in $\mathcal{S}'(\mathbb{C})/\mathbb{R}$ with the following covariance for all $f, h \in \mathcal{S}_0(\mathbb{C})$:

$$\mathbb{E}[(\int_{\mathbb{C}} f(x)\overline{X}(x)d^2x)(\int_{\mathbb{C}} h(y)\overline{X}(y)d^2y)] = \int_{\mathbb{C}^2} f(x)h(y)\ln\frac{1}{|x-y|}d^2xd^2y.^{2} \tag{1.2.2}$$

With this full plane GFF we can define a GFF living in $\mathcal{S}'(\mathbb{C})$ by prescribing the value of $\overline{X}$ against a probability measure on $\mathbb{C}$. This will fix the undetermined constant and thus X will be defined against any test function in $\mathcal{S}'(\mathbb{C})$. Anticipating the coming constructions of LCFT it is natural to write $g(x)d^2x$ for the measure on $\mathbb{C}$ with which we define:

$$X_g(x) = \overline{X}(x) - \int_{\mathbb{C}} \overline{X}(x)g(x)d^2x. \tag{1.2.3}$$

[2]This is a suitable definition for the covariance of a Gaussian process as one can prove by Fourier analysis the following identity,

$$\int_{\mathbb{C}^2} f(x)h(y)\ln\frac{1}{|x-y|}d^2xd^2y = c\int_{\mathbb{C}} \frac{\hat{f}(\xi)\overline{\hat{h}(\xi)}}{|\xi|^2}d^2\xi > 0, \tag{1.2.2}$$

where $c > 0$ is a constant.

We will say that X_g is the GFF on the Riemann sphere $\mathbb{S}^2$ of zero average with respect to the metric g. A natural choice is to chose,

$$\hat{g}(x) := \frac{4}{(1+|x|^2)^2}, \tag{1.2.4}$$

as this is the canonical spherical metric on $\mathbb{S}^2$. More explicitly with this choice $\hat{g}$ the field $X_{\hat{g}}$ then has the covariance:

$$\mathbb{E}[X_{\hat{g}}(x)X_{\hat{g}}(y)] = \ln\frac{1}{|x-y|} - \frac{1}{4}\ln\hat{g}(x) - \frac{1}{4}\ln\hat{g}(y) + \ln 2. \tag{1.2.5}$$

This completes the description of the different GFF on the plane and on the sphere.

We move on to the boundary case and start by looking at the unit disk $\mathbb{D}$. When one works on a domain with boundary there is of course a choice to be made of which boundary conditions to work with. The most natural choice one can think of are the Dirichlet boundary conditions where one imposes the GFF X to be equal to 0 on the boundary. It turns out that this is not the right choice to construct LCFT as this boundary condition is much too strong and many of the expected properties of CFT - see section 1.4 - would fail to hold. Hence we will rather choose Neumann boundary conditions, also sometimes called free boundary conditions. The covariance of the Neumann GFF X is given for $x, y \in \mathbb{D}$ by:

$$\mathbb{E}[X(x)X(y)] = \ln\frac{1}{|x-y||1-x\overline{y}|}. \tag{1.2.6}$$

Of course just like for the case of the full plane GFF, X lives in the space of distributions so the equation (1.2.6) is to be understood by,

$$\mathbb{E}[(\int_{\mathbb{D}} f(x)X(x)d^2x)(\int_{\mathbb{D}} h(y)X(y)d^2y)] = \int_{\mathbb{D}^2} f(x)h(y)\ln\frac{1}{|x-y||1-x\overline{y}|}d^2xd^2y, \tag{1.2.7}$$

where again f and h are smooth test functions defined on $\mathbb{D}$. With this covariance established one can then give the covariance for the GFF defined on $\mathbb{H}$ by using the conformal mapping $z \to \frac{z-i}{z+i}$ linking $\mathbb{H}$ and $\mathbb{D}$. This provides the expression:

$$\mathbb{E}[X_{\mathbb{H}}(x)X_{\mathbb{H}}(y)] = \ln\frac{1}{|x-y||x-\overline{y}|} + \ln|x+i|^2 + \ln|y+i|^2 - 2\ln 2. \tag{1.2.8}$$

By using similar techniques one can also write down the covariance for the Neumann boundary GFF on the annulus $\Omega = \{x, 1 < |x| < \tau\}$. For $x, y \in \Omega$ we have,

$$\mathbb{E}[X_\Omega(x)X_\Omega(y)] = \tilde{G}(x,y) + \ln\frac{|\tau^4x^2y^2|}{|1-x\overline{y}||\tau^2-x\overline{y}||x-y||\tau^2x-y|}, \tag{1.2.9}$$

where $\tilde{G}(x,y)$ is some continuous function and where the factor $|\tau^2x-y|$ holds for $|x|<|y|$ and is replaced by $|x-\tau^2y|$ for $|x|>|y|$. Thus up to a continuous part we observe a similar structure as for the case of the unit disk $\mathbb{D}$.

Lastly we conclude this discussion with the one dimensional cases that we need, the log-correlated fields in both cases of the unit circle and of the unit interval. For the case of the circle $\partial\mathbb{D}$ we will simply restrict the Neumann boundary GFF X defined on the disk $\mathbb{D}$ to the circle. The covariance we thus obtain is given for two points $e^{i\theta}$, $e^{i\theta'}$ by:

$$\mathbb{E}[X(e^{i\theta})X(e^{i\theta'})] = 2\ln\frac{1}{|e^{i\theta}-e^{i\theta'}|}. \tag{1.2.10}$$

For the case of the unit interval $[0,1]$, by analogy with (1.2.10) we will simply chose for our field X_I defined on $[0,1]$ the covariance given for $x, y \in [0,1]$ by:

$$\mathbb{E}[X_I(x)X_I(y)] = 2\ln\frac{1}{|x-y|}. \tag{1.2.11}$$

Both of these one-dimensional fields can also be described quite simply by a Fourier basis expansion. In the case of the unit circle $\partial\mathbb{D}$, let α_n, β_n be two sequences of i.i.d. $\mathcal{N}(0,1)$ random variables. Then we can define X on $\partial\mathbb{D}$ by,

$$\int_0^{2\pi} f(e^{i\theta})X(e^{i\theta})d\theta = \int_0^{2\pi} f(e^{i\theta})\sum_{n\geq 1}\sqrt{\frac{2}{n}}(\alpha_n\cos(n\theta)+\beta_n\sin(n\theta))d\theta, \tag{1.2.12}$$

where again f is a suitable test function. Similarly for the case of the unit interval one can also write down a similar expansion that will involve this time the Chebyshev polynomials.

1.2.2 Gaussian multiplicative chaos measures

We will now introduce the Gaussian multiplicative chaos measures, a fundamental building block of LCFT. Our goal is to define a random measure with density with respect to the Lebesgue measure given by the exponential of our log-correlated fields. The major problem that must be overcome comes from the fact that the exponential of a distribution is ill-defined. Thus this requires to use a regularization and renormalization procedure. The same method will work to define GMC measures associated to all the different cases of GFF that we have described above. We chose to present the technique first for the case of the GFF X on the unit disk $\mathbb{D}$. We discuss both bulk and boundary GMC measures.

The first step is to provide a regularization procedure of X, meaning that we look for a X_ϵ that will be a smooth function for all $\epsilon > 0$ and that will converge to X when ϵ goes to 0. A possible method to do this is to use circle average regularization although many other paths are possible. For $\epsilon > 0$ we call $l_\epsilon(x)$ the length of the arc $A_\epsilon(x) = \{z \in \mathbb{D}; |z-x| = \epsilon\}$ and we set:

$$X_\epsilon(x) = \frac{1}{l_\epsilon(x)}\int_{A_\epsilon(x)} X(x+s)ds. \tag{1.2.13}$$

For a point x at a distance larger than ϵ from the boundary this definition gives the standard circle average. When x is too close to the boundary or on the boundary then we restrict the average to the portion of the circle that remains contained in $\mathbb{D}$. By a direct computation on the covariance (1.2.6) we have the following result:

Lemma 1.2.1. *As $\epsilon \to 0$ we have the following convergences.*
1) Uniformly over all compact subsets of $\mathbb{D}$:

$$\mathbb{E}[X_\epsilon(x)^2] + \ln\epsilon \underset{\epsilon\to 0}{\to} \frac{1}{2}\ln g_P(x)$$

2) Uniformly over all compact subsets of $\partial\mathbb{D}$:

$$\mathbb{E}[X_\epsilon(e^{i\theta})^2] + 2\ln\epsilon \underset{\epsilon\to 0}{\to} -1$$

Here $g_P(x) = \frac{1}{|1-|x|^2|^2}$ is the so-called Poincaré hyperbolic metric.

We now state the following proposition-definition of Gaussian multiplicative chaos:

Proposition 1.2.2. *(Definition of GMC on $\mathbb{D}$) Let $\gamma \in (0,2)$. We then define the random measures $e^{\gamma X(x)}d^2x$ and $e^{\frac{\gamma}{2}X(e^{i\theta})}d\theta$ as the following limits in probability*

$$e^{\gamma X(x)}d^2x := \lim_{\epsilon\to 0}\epsilon^{\frac{\gamma^2}{2}}e^{\gamma X_\epsilon(x)}d^2x = \lim_{\epsilon\to 0}e^{\gamma X_\epsilon(x)-\frac{\gamma^2}{2}\mathbb{E}[X_\epsilon(x)^2]}g_P(x)^{\frac{\gamma^2}{4}}d^2x \tag{1.2.14}$$

$$e^{\frac{\gamma}{2}X(e^{i\theta})}d\theta := \lim_{\epsilon\to 0}\epsilon^{\frac{\gamma^2}{4}}e^{\frac{\gamma}{2}X_\epsilon(e^{i\theta})}d\theta = \lim_{\epsilon\to 0}e^{\frac{\gamma}{2}X_\epsilon(e^{i\theta})-\frac{\gamma^2}{8}\mathbb{E}[X_\epsilon(e^{i\theta})^2]}e^{-\frac{\gamma^2}{8}}d\theta \tag{1.2.15}$$

in the sense of weak convergence of measures respectively over $\mathbb{D}$ and $\partial\mathbb{D}$. More precisely this means that for all continuous test functions $f:\mathbb{D}\to\mathbb{R}$ the following convergences hold in probability,

$$\int_{\mathbb{D}} f(x)e^{\gamma X(x)}d^2x := \lim_{\epsilon\to 0}\epsilon^{\frac{\gamma^2}{2}}\int_{\mathbb{D}} f(x)e^{\gamma X_\epsilon(x)}d^2x, \tag{1.2.16}$$

and

$$\int_{\partial\mathbb{D}} f(e^{i\theta})e^{\frac{\gamma}{2}X(e^{i\theta})}d\theta := \lim_{\epsilon\to 0}\epsilon^{\frac{\gamma^2}{4}}\int_{\partial\mathbb{D}} f(e^{i\theta})e^{\frac{\gamma}{2}X_\epsilon(e^{i\theta})}d\theta. \tag{1.2.17}$$

To have a finite partition function for LCFT, we must show that these GMC measures have an almost surely finite total mass. It is indeed shown in [40] that:

Proposition 1.2.3. *For $\gamma \in (0,2)$ the following quantities are almost surely finite:*

$$\int_{\mathbb{D}} e^{\gamma X(x)}d^2x \quad \text{and} \quad \int_{\partial\mathbb{D}} e^{\frac{\gamma}{2}X(e^{i\theta})}d\theta.$$

Of course a similar procedure can be used to define GMC on the Riemann sphere. Introducing first a regularization of X_g,

$$X_{g,\epsilon}(x) = \frac{1}{2\pi}\int_0^{2\pi} X_g(x+\epsilon e^{i\theta})d\theta, \tag{1.2.18}$$

we can perform the same computation of the one of lemma 1.2.1 and write down a similar proposition to define GMC:

Proposition 1.2.4. *(Definition of GMC on $\mathbb{S}^2$) Let $\gamma \in (0,2)$. We then define the random measure $e^{\gamma X_g(x)}g(x)d^2x$ as the following limit in probability*

$$e^{\gamma X_g(x)}g(x)d^2x := \lim_{\epsilon\to 0}e^{\gamma X_{g,\epsilon}(x)-\frac{\gamma^2}{2}\mathbb{E}[X_{g,\epsilon}(x)^2]}g(x)d^2x \tag{1.2.19}$$

in the sense of weak convergence of measures over $\mathbb{S}^2$.

1.2.3 Moments of GMC measures

In this short subsection we give a list of results on the existence of moments of GMC on $\mathbb{S}^2$, $\mathbb{D}$, and $\partial\mathbb{D}$. We also include the generalization where certain fractional moments are added in the GMC measure. The bounds obtained will be extremely important for the definitions of the correlations of LCFT. We start with the existence of moments:

Proposition 1.2.5. *(Moments of GMC on $\mathbb{S}^2$) Let $\hat{g}$ be the canonical spherical metric on $\mathbb{S}^2$ and let $\gamma \in (0,2)$. Then we have,*

$$\mathbb{E}[(\int_{\mathbb{C}} e^{\gamma X_{\hat{g}}(x)}\hat{g}(x)d^2x)^p] < +\infty, \tag{1.2.20}$$

if and only if $p < \frac{4}{\gamma^2}$.

We also provide a similar result for $\mathbb{D}$ and $\partial\mathbb{D}$:

Proposition 1.2.6. *(Moments of GMC on $\mathbb{D}$) Let $\gamma \in (0,2)$, then we have*

$$\mathbb{E}[(\int_{\mathbb{D}} e^{\gamma X(x)}d^2x)^p] < +\infty \tag{1.2.21}$$

if and only if $p < \frac{2}{\gamma^2}$ and

$$\mathbb{E}[(\int_{\partial\mathbb{D}} e^{\frac{\gamma}{2}X(e^{i\theta})}d\theta)^p] < +\infty \tag{1.2.22}$$

if and only if $p < \frac{4}{\gamma^2}$.

We have a different result for the moments of the bulk measure because of the Poincaré metric g_P and because of the behaviour of the GFF X near the boundary. To study LCFT we also need the same type of results but with insertion points. The bounds for non-triviality of a GMC moment with insertion have been obtained in [17, 40], they are:

Proposition 1.2.7. *(Moments of GMC with insertions) Let $\gamma \in (0,2)$, $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$, $\alpha, \beta \in \mathbb{R}$, $z \in \mathbb{C}$ with $|z| < 1$, and $s \in \partial\mathbb{D}$. Then we have,*

$$0 < \mathbb{E}[(\int_{\mathbb{C}} \frac{1}{|z-x|^{\alpha\gamma}} e^{\gamma X_{\hat{g}}(x)}\hat{g}(x)d^2x)^p] < +\infty, \tag{1.2.23}$$

if and only if $\alpha < Q$ and $p < \frac{4}{\gamma^2} \wedge \frac{2}{\gamma}(Q-\alpha)$. Similarly,

$$0 < \mathbb{E}[(\int_{\mathbb{D}} \frac{1}{|z-x|^{\alpha\gamma}} \frac{1}{|s-x|^{\frac{\beta\gamma}{2}}} e^{\gamma X(x)}d^2x)^p] < +\infty, \tag{1.2.24}$$

if and only if[3] $\alpha < Q$, $\beta < Q$, and $p < \frac{2}{\gamma^2} \wedge \frac{2}{\gamma}(Q-\alpha) \wedge \frac{1}{\gamma}(Q-\beta)$. Lastly,

$$0 < \mathbb{E}[(\int_{\partial\mathbb{D}} \frac{1}{|s-e^{i\theta}|^{\frac{\beta\gamma}{2}}} e^{\frac{\gamma}{2}X(e^{i\theta})}d\theta)^p] < +\infty, \tag{1.2.25}$$

if and only if $\beta < Q$ and $p < \frac{4}{\gamma^2} \wedge \frac{2}{\gamma}(Q-\beta)$.

[3] For $\gamma \in (\sqrt{2}, 2)$ it remains to be proved that the bound written above on p is optimal although it is strongly believed to be the case, see [40] for an explanation of this subtlety.

1.3 Liouville conformal field theory

We now move on to the study of Liouville conformal field theory (LCFT). As we mentioned in introduction this theory first appeared in Polyakov's seminal 1981 paper [66] out of the need to understand what is a canonical random Riemannian metric on a surface of given topology. More precisely Polyakov tells us that for a fixed reference metric g on a given surface M one should consider the random metric $e^{\gamma\phi}g$ where ϕ is the Liouville field.

This section will provide a self-contained construction of LCFT on different surfaces. Based on the topology of the surface M there will be some differences in the construction. First we will explain the simplest case, LCFT on the Riemann sphere following [17]. Next will come the case of the unit disk $\mathbb{D}$ first studied in [40] where one has to cope with the presence of a boundary. Lastly we will consider the case of the annulus Ω, a domain with two boundaries and with a non-trivial moduli space. The detailed study of this case is precisely the content of chapter 4. Let us mention that LCFT has also been constructed on other domains with a non-trivial moduli space, see [18] for the torus and [37] for compact surfaces of higher genus.

1.3.1 LCFT on the Riemann sphere

The case of the Riemann sphere $\mathbb{S}^2$ studied in detail in [17] is the simplest to define the theory of LCFT as it is a compact simply connected boundaryless surface. Our choice of coordinates to represent the sphere $\mathbb{S}^2$ will be the complex plane with a point added at infinity $\mathbb{C} \cup \{\infty\}$.

To define LCFT, physicists use what is referred to as the path integral formalism.[4] Informally it tells us that our Liouville field ϕ will be given in terms of an infinite measure on a suitable functional space. Until the very recent work [17] giving a rigorous probabilistic content to LCFT had remained an open problem. We now sketch the physicists' heuristic definitions before explaining how to make them rigorous. Consider the following space of maps:

$$\Sigma = \{X : \mathbb{S}^2 \to \mathbb{R}\}. \tag{1.3.1}$$

The Liouville field ϕ is then given by the following formal definition, for any background metric g on $\mathbb{S}^2$,

$$\mathbb{E}[F(\phi)] = \frac{1}{\mathcal{Z}} \int_\Sigma F(X) e^{-S_L(X,g)} DX, \tag{1.3.2}$$

where $S_L(X,g)$ is the so-called Liouville action:

$$S(X,g) = \frac{1}{4\pi} \int_{\mathbb{S}^2} (|\partial^g X|^2 + QR_g X + 4\pi\mu e^{\gamma X}) g(x) d^2x. \tag{1.3.3}$$

Let us comment on all our notations. The three real parameters γ, Q, μ that appear obey $\gamma \in (0,2)$, $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$, and $\mu > 0$. DX is the formal uniform measure on the space Σ and g is the background Riemannian metric used to define the theory. To avoid going too deep into the framework of Riemannian geometry we restrict ourselves to the case where the background metric g is a diagonal

[4] It is instructive to note that the path integral formalism we explain can also be used to heuristically define Brownian motion. Consider the space of path $\Sigma' = \{\sigma : [0,1] \to \mathbb{R},\ \sigma(0) = 0\}$ and the action $S_{BM} = \frac{1}{2}\int_0^1 |\sigma'(t)|^2 dt$. Then for all suitable F,

$$\mathbb{E}[F((B_s)_{0 \le s \le 1})] = \frac{1}{\mathcal{Z}} \int_{\Sigma'} F(\sigma) e^{-S_{BM}(\sigma)} D\sigma,$$

where $D\sigma$ is a formal uniform measure on Σ'.

tensor, meaning that g will simply be a positive function defined on $\mathbb{C} \cup \{\infty\}$. In this simple case we have $|\partial^g X|^2 = g^{-1}|\partial X|^2$ and $R_g = -g^{-1}\Delta \ln g$. As a matter of fact it turns out that on the Riemman sphere up to a change of coordinates any metric tensor can be written under this form. Lastly $\mathcal{Z}$ is a formal normalization constant.

Let us now explain the three terms appearing in the Liouville action. The gradient term $|\partial^g X|^2$ of (1.3.3) is the free field or kinetic energy term. If this was the only term present, - up to a global constant - the law of ϕ would be that of our Gaussian free field X_g on $\mathbb{S}^2$. But the action (1.3.3) also contains a non-linear interaction term, the exponential term $e^{\gamma X}$. At first glance it may seem unclear why it is interesting to consider this specific interaction. The first motivation comes from theoretical physics - in particular string theory and 2d quantum gravity, see [16, 20, 21, 66] - where as we have said earlier $e^{\phi}g$ formally defines the correct random Riemannian metric on $\mathbb{S}^2$. The second reason is that, out of all possible interaction terms, the exponential term is the simplest term that defines a conformal field theory. The implications of this will be explained thoroughly in sections 1.4 and 1.5. Finally, the term QR_gX is the linear coupling of X to the background metric g. It is required to get a consistent theory but does not pose any mathematical problems as it is a linear term in X.

One may wonder why the Liouville action is the correct action to define canonical random metrics. A first answer comes from physics, in particular from Polyakov in [66], see the appendix of chapter 4 for a more detailed discussion. An easier answer comes from the study of classical Liouville theory, meaning that we look for the functions X minimizing the Liouville action. It is a well known fact of classical geometry that such a minimum $X_{\min}$ is unique if it exists and the new metric $g' = e^{\gamma X_{\min}} g$ is of constant negative curvature provided that $Q_c = \frac{2}{\gamma}$[5]. In other words, the minimum of the Liouville action uniformizes the surface (M, g) and it is therefore natural to look at quantum fluctuations of the uniformized metric $e^{\gamma X_{\min}} g$. This is precisely the meaning of (1.3.2).

As we have written it the path integral (1.3.2) diverges for any surface of genus 0 or 1, including therefore the case of the sphere. To see this we can write the Gauss-Bonnet formula, given here for a boundaryless surface M of genus h equipped with a metric g, $\int_M R_g(x)g(x)d^2x = 8\pi(1-h)$. When $h = 0$ or 1, this implies that it is impossible to define on the surface a metric of constant negative curvature, meaning that $S_L(X, g)$ will have no minimum and therefore the path integral (1.3.2) diverges. To solve this problem we proceed as in [17] and add insertion points. We consider the new expression,

$$\mathbb{E}[F(\phi)] = \frac{1}{\mathcal{Z}} \int_{\Sigma} F(X) e^{\sum_{i=1}^N \alpha_i X(z_i)} e^{-S_L(X,g)} DX, \tag{1.3.4}$$

where we have chosen N insertion points $z_i \in \mathbb{S}^2$ with weights $\alpha_i \in \mathbb{R}$. By choosing $F = 1$ in the above expression we define the N-point correlation function of LCFT, the most fundamental observable of the theory:

$$\langle \prod_{i=1}^N e^{\alpha_i \phi(z_i)} \rangle_{\mathbb{S}^2, g} = \frac{1}{\mathcal{Z}} \int_{\Sigma} e^{\sum_{i=1}^N \alpha_i X(z_i)} e^{-S_L(X,g)} DX. \tag{1.3.5}$$

We show that the following conditions known as the Seiberg bounds must be satisfied in order

[5]Here we write $Q_c = \frac{2}{\gamma}$ as this is the correct value in the classical theory where the goal is to minimize the Liouville action. In the quantum theory we will always have $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$.

for (1.3.4) and (1.3.5) to exist:

$$\sum_{i=1}^{N} \alpha_i > 2Q \quad \text{and} \quad \forall i,\ \alpha_i < Q. \tag{1.3.6}$$

The minimum number of insertion points needed to satisfy these bounds is three. This is precisely the requirement to entirely determine a conformal automorphism of the sphere, the so-called Mobius transformations. From a geometric standpoint, we can also view insertion points as conical singularities of the metric which allow hyperbolic metrics to be defined on the surface.

We now move on to the probabilistic definition of (1.3.5). This will require to use both the Gaussian free field and the Gaussian multiplicative chaos measures. The GFF will appear out of the need to make sense of the formal density $e^{-\frac{1}{4\pi}\int_{\mathbb{C}}|\partial^g X|^2 g(x)d^2x}DX$ which we will interpret as the density of a Gaussian vector in infinite dimensions. The key observation is that by performing an integration by parts we have:

$$\frac{1}{4\pi}\int_{\mathbb{C}} |\partial^g X|^2 g(x) d^2x = \frac{1}{2}\int_{\mathbb{C}} X\left(-\frac{\Delta_g}{2\pi}\right) X g(x) d^2x. \tag{1.3.7}$$

Therefore we are formally constructing a field with covariance given by the inverse of the Laplacian. To give a more precise mathematical meaning to this observation we will replace the abstract space Σ by the following L^2 space,

$$L^2(\mathbb{S}^2) := \{X, \int_{\mathbb{C}} X(x)^2 g(x) d^2x < +\infty\}, \tag{1.3.8}$$

on which we can diagonalize the Laplacian. Now let $(e_j)_{j\geq 1}$ be a basis of eigenvectors for $-\Delta_g$ meaning that,

$$-g(x)^{-1}\Delta e_j(x) = \lambda_j e_j(x), \tag{1.3.9}$$

where the e_j are normalized to have an L^2 norm equal to 1: $\int_{\mathbb{C}} e_j(x)^2 g(x) d^2x = 1$. With this basis any function $X \in L^2(\mathbb{S}^2)$ can be written,

$$X = c + \sum_{j=1}^{\infty} c_j e_j, \tag{1.3.10}$$

where the coefficient c_j are obtained by $c_j = \int_{\mathbb{C}} X(x) e_j(x) g(x) d^2x$. Such a decomposition tells us that we have,

$$\frac{1}{4\pi}\int_{\mathbb{C}} |\partial^g X|^2 g(x) d^2x = \frac{1}{4\pi}\sum_{j=1}^{\infty} c_j^2 \lambda_j, \tag{1.3.11}$$

and so heuristically it is natural to want to write:

$$\int_{L^2(\mathbb{S}^2)} F(X) e^{-\frac{1}{4\pi}\int_{\mathbb{C}}|\partial^g X|^2 g(x)d^2x} DX = \int_{\mathbb{R}}\int_{\mathbb{R}^{\mathbb{N}^*}} F(c + \sum_{j=1}^{\infty} c_j e_j) dc \prod_{j=1}^{\infty} e^{-\frac{c_j^2\lambda_j}{4\pi}} dc_j. \tag{1.3.12}$$

Here dc and each dc_i are Lebesgue measures on $\mathbb{R}$. The sum $\sum_{j\geq 1}\sqrt{\frac{2\pi}{\lambda_j}}\epsilon_j e_j(x)$ with $(\epsilon_j)_{j\geq 1}$ being a sequence of i.i.d. standard Gaussians converges in the space of distributions towards X_g and thus we can make sense of the above formal expression by writing that:

$$\int_{L^2(\mathbb{S}^2)} F(X) e^{-\frac{1}{4\pi}\int_{\mathbb{C}}|\partial^g X_g|^2 g(x)d^2x} DX := \int_{\mathbb{R}} \mathbb{E}[F(X_g + c)] dc. \tag{1.3.13}$$

In the above expression, the left hand side is a formal expression where X is an integration variable. On the right hand side X_g is the GFF that we have constructed. By construction of our GFF X_g on $\mathbb{S}^2$ we have $\int_{\mathbb{C}} X_g(x)R_g(x)g(x)d^2x = 0$. Therefore:

$$\int_{L^2(\mathbb{S}^2)} F(X)e^{-\frac{1}{4\pi}\int_{\mathbb{C}}|\partial^g X|^2 g(x)d^2x - \frac{1}{4\pi}\int_{\mathbb{C}} X(x)R_g(x)g(x)d^2x} DX := \int_{\mathbb{R}} e^{-2Qc}\mathbb{E}[F(X_g + c)]dc. \quad (1.3.14)$$

Thus using the Gaussian free field we have given a probabilistic meaning to (1.3.2) in the very special case where $\mu = 0$. We now tackle the problem of handling the exponential interaction term, this is where we will require the theory of Gaussian multiplicative chaos. To define the correlations (1.3.5) we want to pick in (1.3.14) a functionnal F of the type,

$$F(X) = \prod_{i=1}^{N} e^{\alpha_i X(z_i)} e^{-\mu\int_{\mathbb{C}} e^{\gamma X(x)}g(x)d^2x}, \quad (1.3.15)$$

but since X_g is not defined pointwise but lives in the space of distributions this will require a regularization and renormalization procedure. We will use again the notation $X_{g,\epsilon}(x)$ for the circle average regularization of X_g. We now state the theorem that insures that the correlations of LCFT on $\mathbb{S}^2$ are well-defined by using the regularization-renormalization procedure.

Theorem 1.3.1. *(Correlation functions of LCFT on $\mathbb{S}^2$) Choose N points $z_i \in \mathbb{S}^2$ of weights $\alpha_i \in \mathbb{R}$ such that the Seiberg bounds (1.3.6) are satisfied. Then the N-point correlation function of LCFT is defined by the following limit,*

$$\langle\prod_{i=1}^{N} e^{\alpha_i\phi(z_i)}\rangle_{\mathbb{S}^2,g} := \lim_{\epsilon\to 0}\langle\prod_{i=1}^{N} e^{\alpha_i\phi(z_i)}\rangle_{\mathbb{S}^2,g,\epsilon}, \quad (1.3.16)$$

where the expresssion of the regularized correlation is given by:

$$\langle\prod_{i=1}^{N} e^{\alpha_i\phi(z_i)}\rangle_{\mathbb{S}^2,g,\epsilon} := \int_{\mathbb{R}} e^{-2Qc}\mathbb{E}[\prod_{i=1}^{N} \epsilon^{\frac{\alpha_i^2}{2}} e^{\alpha_i(X_{g,\epsilon}(z_i)+\frac{Q}{2}\ln g(z_i)+c)} \exp(-\mu e^{\gamma c}\epsilon^{\frac{\gamma^2}{2}}\int_{\mathbb{C}} e^{\gamma(X_{g,\epsilon}(x)+\frac{Q}{2}\ln g(x))}d^2x)]dc. \quad (1.3.17)$$

The limit written above exists and is different from 0 and $+\infty$. By simple manipulations on the above definition, one obtains the more compact expression,

$$\langle\prod_{i=1}^{N} e^{\alpha_i\phi(z_i)}\rangle_{\mathbb{S}^2,g} = c_g\mu^{-s}\Gamma(s)\prod_{i<j}\frac{1}{|z_i-z_j|^{\alpha_i\alpha_j}}\mathbb{E}[(\int_{\mathbb{C}}\prod_{i=1}^{N}\frac{1}{|x-z_i|^{\gamma\alpha_i}}\tilde{c}_g(x)e^{\gamma X_g(x)}d^2x)^{-s}], \quad (1.3.18)$$

with $s = \frac{\sum_{i=1}^N \alpha_i - 2Q}{\gamma}$ and where c_g and $\tilde{c}_g(x)$ are terms depending on g.

There are two simple steps to go from (1.3.17) to (1.3.18). The first is to apply the Girsanov theorem (also called the complete the square trick in physics) to the insertions $e^{\alpha_i X_{g,\epsilon}(z_i)}$, which produces a shift on the field $X(x) \to X(x) + \sum_i \alpha_i\mathbb{E}[X(x)X(z_i)]$. By then using the explicit expression of the covariance this creates the fractional powers in the GMC measure. The second step is simply to perform a change of variable on the integral over c by setting $u = \mu e^{\gamma c}\int_{\mathbb{C}} e^{\gamma(X_{g,\epsilon}(x)+\frac{Q}{2}\ln g(x))}d^2x$. The integral over u then gives a Gamma function and we are left with (1.3.18). The terms c_g, $\tilde{c}_g(x)$ can be easily computed and play little role in the following, see [17]. This last expression we obtain for the value of the N-point correlation of LCFT on $\mathbb{S}^2$ is fundamental. It tells us that we have a definition for the correlation function in terms of a relatively simple probabilistic object, a moment of a GMC measure with prescribed log-singularities. With this expression well-established one can now hope to implement rigorously in a probabilistic framework all the techniques of conformal field theory.

1.3.2 LCFT on a domain with boundary

Following [40] LCFT can of course also be defined on a domain with a boundary. We start by explaining the construction on the unit disk $\mathbb{D}$ and we denote by $\partial\mathbb{D}$ its boundary. By conformal mapping this will provide as well the construction of LCFT on $\mathbb{H}$. The biggest difference with the case of the Riemann sphere $\mathbb{S}^2$ is that we can now have two exponential interaction terms, a bulk interaction $\mu e^{\gamma X} d^2x$ and a boundary interaction $\mu_\partial e^{\frac{\gamma}{2}X} d\theta$. Therefore the Liouville action will contain in this case boundary terms:

$$S_\partial(X, dx^2) = \frac{1}{4\pi}\int_{\mathbb{D}} (|\partial X|^2 + 4\pi\mu e^{\gamma X}) d^2x + \frac{1}{2\pi}\int_{\partial\mathbb{D}} (QX + 2\pi\mu_\partial e^{\frac{\gamma}{2}X}) d\theta. \tag{1.3.19}$$

In the same way as before we can write a heuristic path integral definition for the correlation functions of the theory:

$$\langle \prod_{i=1}^N e^{\alpha_i \phi(z_i)} \prod_{j=1}^M e^{\frac{\beta_j}{2}\phi(s_j)} \rangle_{\mathbb{D}} = \frac{1}{\mathcal{Z}} \int_\Sigma e^{\sum_i \alpha_i X(z_i) + \frac{1}{2}\sum_j \beta_j X(s_j)} e^{-S_\partial(X, dx^2)} DX \tag{1.3.20}$$

In our presentation we have restricted ourselves to the case of the Euclidean metric dx^2 on $\mathbb{D}$ but of course one could also define the theory in any background metric g. We note that there are two types of insertion points, bulk insertions $z_i \in \mathbb{D}$ with weights $\alpha_i \in \mathbb{R}$ that are inside the disk and boundary insertions $s_j \in \partial\mathbb{D}$ with weights $\beta_j \in \mathbb{R}$ that are placed on the unit circle. Similar techniques as the ones explained for the case of $\mathbb{S}^2$ allow one to give a probabilistic meaning to the above path integral. The Seiberg bounds for this case are given by:

$$\sum_{i=1}^N \alpha_i + \sum_{j=1}^M \frac{\beta_j}{2} > Q \quad \text{and} \quad \forall i,\ \alpha_i < Q,\ \forall j,\ \beta_j < Q. \tag{1.3.21}$$

Again we notice that these bounds impose a minimum number of points that corresponds exactly to completely determining a conformal automorphism of the disk $\mathbb{D}$. We can now state:

Theorem 1.3.2. *(Correlation functions of LCFT on $\mathbb{D}$) Choose N points $z_i \in \mathbb{D}$ and M points $s_j \in \partial\mathbb{D}$ such that the bounds (1.3.21) hold. Then the correlation function has an expression given by,*

$$\begin{aligned}\langle \prod_{i=1}^N e^{\alpha_i \phi(z_i)} \prod_{j=1}^M e^{\frac{\beta_j}{2}\phi(s_j)} \rangle_{\mathbb{D}} = \prod_{i=1}^N g_P(z_i)^{\frac{\alpha_i^2}{4}} e^{C(z,s)} \int_{\mathbb{R}} e^{(\sum_i \alpha_i + \sum_j \frac{\beta_j}{2} - Q)c} \\ \times \mathbb{E}[\exp(-\mu e^{\gamma c}\int_{\mathbb{D}} e^{\gamma X + \gamma H} d^2x - \mu_\partial e^{\frac{\gamma c}{2}} \int_{\partial\mathbb{D}} e^{\frac{\gamma}{2}X + \frac{\gamma}{2}H} d\theta)] dc,\end{aligned}$$

where we have used the notations:

$$G(x,y) := \mathbb{E}[X(x)X(y)] = \ln \frac{1}{|x-y||1-x\overline{y}|},$$
$$H(x) := \sum_i \alpha_i G(x,z_i) + \sum_j \frac{\beta_j}{2} G(x,s_j),$$
$$C(z,s) := \sum_{i<i'} \alpha_i \alpha_{i'} G(z_i,z_{i'}) + \sum_{j,j'} \frac{\beta_j \beta_{j'}}{4} G(s_j,s_{j'}) + \sum_{i,j} \frac{\alpha_i \beta_j}{2} G(z_i,s_j) - \sum_j \frac{\beta_j^2}{8}.$$

The difference with $\mathbb{S}^2$ is that if both interaction terms are present, i.e. $\mu > 0$, $\mu_\partial > 0$, then we will not land on an expression as simple as (1.3.18). Indeed, it is not possible to perform the same change of variable on c and we must stick to the expression of Theorem 1.3.2. On the other hand if one chooses $\mu = 0$ or $\mu_\partial = 0$ then the above can be simplified to obtain an expression involving a moment of GMC on the circle $\partial\mathbb{D}$ or respectively on the disk $\mathbb{D}$. In the case $\mu_\partial = 0$ we introduce $Z_0 = \int_{\mathbb{D}} e^{\gamma(X(x)+H(x))} d^2x$, $u = \mu e^{\gamma c} Z_0$, $du = \gamma u dc$. Then one has:

$$\begin{aligned}
\langle \prod_{i=1}^N e^{\alpha_i \phi(z_i)} \prod_{j=1}^M e^{\frac{\beta_j}{2}\phi(s_j)} \rangle_{\mathbb{D},\mu_\partial=0} &= \prod_{i=1}^N g_P(z_i)^{\frac{\alpha_i^2}{4}} e^{C(z,s)} \int_{\mathbb{R}} dc \mathbb{E}[e^{c(\sum_i \alpha_i + \sum_j \frac{\beta_j}{2} - Q)} e^{-\mu e^{\gamma c} Z_0}] \\
&= \prod_{i=1}^N g_P(z_i)^{\frac{\alpha_i^2}{4}} e^{C(z,s)} \int_0^\infty du \frac{1}{\gamma u} \mathbb{E}[(\frac{u}{\mu Z_0})^{\frac{\sum_i \alpha_i + \sum_j \frac{\beta_j}{2} - Q}{\gamma}} e^{-u}] \\
&= \frac{1}{\gamma} \prod_{i=1}^N g_P(z_i)^{\frac{\alpha_i^2}{4}} e^{C(z,s)} \mu^{-\frac{\sum_i \alpha_i + \sum_j \frac{\beta_j}{2} - Q}{\gamma}} \Gamma(\frac{\sum_i \alpha_i + \sum_j \frac{\beta_j}{2} - Q}{\gamma}) \\
&\quad \times \mathbb{E}\left[Z_0^{-\frac{\sum_i \alpha_i + \sum_j \frac{\beta_j}{2} - Q}{\gamma}} \right]
\end{aligned} \tag{1.3.22}$$

In the case $\mu = 0$, we introduce $Z_0^\partial = \int_{\partial\mathbb{D}} e^{\frac{\gamma}{2}(X(e^{i\theta})+H(e^{i\theta}))} d\theta$, $u = \mu_\partial e^{\frac{\gamma c}{2}} Z_0^\partial$, $du = \frac{\gamma u}{2} dc$. Then similarly as before:

$$\begin{aligned}
\langle \prod_{i=1}^N e^{\alpha_i \phi(z_i)} \prod_{j=1}^M e^{\frac{\beta_j}{2}\phi(s_j)} \rangle_{\mathbb{D},\mu=0} &= \prod_{i=1}^N g_P(z_i)^{\frac{\alpha_i^2}{4}} e^{C(z,s)} \int_{\mathbb{R}} dc \mathbb{E}[e^{c(\sum_i \alpha_i + \sum_j \frac{\beta_j}{2} - Q)} e^{-\mu_\partial e^{\frac{\gamma c}{2}} Z_0^\partial}] \\
&= \prod_{i=1}^N g_P(z_i)^{\frac{\alpha_i^2}{4}} e^{C(z,s)} \int_0^\infty du \frac{2}{\gamma u} \mathbb{E}[(\frac{u}{\mu_\partial Z_0^\partial})^{\frac{\sum_i 2\alpha_i + \sum_j \beta_j - 2Q}{\gamma}} e^{-u}] \\
&= \frac{2}{\gamma} \prod_{i=1}^N g_P(z_i)^{\frac{\alpha_i^2}{4}} e^{C(z,s)} \mu_\partial^{-\frac{\sum_i 2\alpha_i + \sum_j \beta_j - 2Q}{\gamma}} \Gamma(\frac{\sum_i 2\alpha_i + \sum_j \beta_j - 2Q}{\gamma}) \\
&\quad \times \mathbb{E}\left[(Z_0^\partial)^{-\frac{\sum_i 2\alpha_i + \sum_j \beta_j - 2Q}{\gamma}} \right].
\end{aligned} \tag{1.3.23}$$

1.3.3 LCFT on the annulus and the moduli space, *Summary of chapter 4*

The last topology we will consider for the construction of LCFT is that of the annulus Ω, a detailed study of this case is presented in chapter 4. An annulus is a domain with two boundaries so the construction of LCFT will be similar to the case of the unit disk $\mathbb{D}$. The major difference is that the annulus posses a non-trivial moduli space. Whereas all simply connected subdomains of $\mathbb{C}$ can be conformally mapped to the unit disk $\mathbb{D}$, all annuli are not equivalent under conformal mapping. Indeed, there exists a conformal map between two annuli of respective radii a, b and a', b' if and only if $\frac{b}{a} = \frac{b'}{a'}$. If this is the case then the common ratio $\tau = \frac{b}{a}$ is called the modulus. The moduli space is then $(1, +\infty)$ and our parameter $\tau \in (1, +\infty)$ parametrizes the moduli space.

We introduce the notation $\Omega = \{z \in \mathbb{C}, 1 < |z| < \tau\}$ for the annulus of radii 1 and τ where $\tau \in (1, +\infty)$. The construction of the Liouville field ϕ on Ω for a fixed τ follows the same ideas as for the case of $\mathbb{D}$ although some steps are much more technical (notice for instance that the expression of the covariance of X_Ω is more convoluted). On the other hand, to construct the full theory of Liouville quantum gravity on Ω, one must integrate the partition function of Liouville theory at fixed τ over the moduli space $(1, +\infty)$ with an appropriate measure. This procedure is described in great detail in chapter 4 where we construct the partition function of Liouville quantum gravity $\mathcal{Z}_{LQG}$ and show that it is finite:

Theorem 1.3.3. *Let us assume that $\mu_\partial > 0$. Then the partition function of Liouville quantum gravity given by*

$$\mathcal{Z}_{LQG} = \int_1^\infty \frac{d\tau}{\tau} \tau^{\frac{c_m - 25}{12}} |\eta(\tau)|^{1-c_m} \mathcal{Z}_{GFF}^{-1} \Pi_{\gamma,\mu,\mu_\partial}^{(1,\gamma)}(\tau, dx^2)$$

is finite for any value of $\gamma \in (0, 2)$.

We refer the reader to chapter 4 for the definition of the quantities appearing in the above theorem. This result allows us to give the joint law of the Liouville bulk measure Z, the Liouville boundary measure Z^∂, and of the random modulus τ. It also allows to give the conjectured link with random planar maps, see the conjecture stated at the end of chapter 4. Lastly we mention that our construction of LCFT on the annulus addresses some points that were left out in [40] and gives simpler proofs of certain results such as the KPZ formula of Theorem 1.4.4 which is deduced directly from Theorem 1.4.2.

1.4 The techniques of conformal field theory

Now that we have at our disposal a probabilistic construction of LCFT we can verify that it satisfies all the different properties expected of a conformal field theory. This will provide extremely powerful techniques to perform exact computations on the Liouville theory. In particular the celebrated equations of Belavin-Polyakov-Zamolodchikov (BPZ) translate the constraints imposed by the local conformal invariance of CFT. By using these equations we will be able to obtain exact computations on certain correlation functions of the theory. We insist on the fact that all the properties we will consider - Weyl anomaly, KPZ formula, Ward identities and BPZ equations - are theorems that are proved starting from the definition given by Theorem 1.3.1. They are not imposed as axioms as in other approaches [71]. In the following three subsections there will always be two cases, the case of the Riemann sphere $\mathbb{S}^2$ and the case of a domain with boundary.

1.4.1 Weyl anomaly and KPZ formula

In section 1.3 we gave a definition of LCFT for any background metric g defined on the Riemann sphere $\mathbb{S}^2$. It is thus natural to wonder how the correlation functions of LCFT depend on this choice of background metric. The answer to this question is given by the Weyl anomaly, a property expected of all CFT's. Following [17]:

Theorem 1.4.1. *(Weyl anomaly on $\mathbb{S}^2$) Given a metric $g = e^{\varphi}\hat{g}$ conformally equivalent to the spherical metric $\hat{g}$ we have,*

$$\langle\prod_{i=1}^N e^{\alpha_i\phi(z_i)}\rangle_{\mathbb{S}^2,g} = \exp\left(\frac{c_L}{96\pi}(\int_{\mathbb{C}}|\partial\varphi(x)|^2d^2x + 4\int_{\mathbb{C}}\varphi(x)\hat{g}(x)d^2x)\right)\langle\prod_{i=1}^N e^{\alpha_i\phi(z_i)}\rangle_{\mathbb{S}^2,\hat{g}}, \tag{1.4.1}$$

where $c_L = 1 + 6Q^2$. This constant c_L is the so-called central charge of the Liouville conformal field theory.

We can also write down the exact same theorem for a domain with boundary. We write here the result for the unit disk $\mathbb{D}$ although of course a similar formula is true for the upper half plane $\mathbb{H}$ or for the annulus Ω. Following [40, 69]:

Theorem 1.4.2. *(Weyl anomaly on $\mathbb{D}$) Given a metric $g = e^{\varphi}dx^2$ conformally equivalent to the Euclidean metric dx^2 on $\mathbb{D}$ we have,*

$$\langle\prod_{i=1}^N e^{\alpha_i\phi(z_i)}\prod_{j=1}^M e^{\frac{\beta_j}{2}\phi(s_j)}\rangle_{\mathbb{D},g} = \exp\left(\frac{c_L}{96\pi}(\int_{\mathbb{D}}|\partial\varphi(x)|^2d^2x + 4\int_{\partial\mathbb{D}}\varphi(e^{i\theta})d\theta)\right)\langle\prod_{i=1}^N e^{\alpha_i\phi(z_i)}\prod_{j=1}^M e^{\frac{\beta_j}{2}\phi(s_j)}\rangle_{\mathbb{D}}, \tag{1.4.2}$$

where again $c_L = 1 + 6Q^2$.

We now move on to the KPZ formula which describes the behaviour of the Liouville theory when one applies a conformal automorphism to the domain. In other words it tells us that the correlations of LCFT behave as conformal tensors. We start with $\mathbb{S}^2$, see [17]:

Theorem 1.4.3. *(KPZ formula on $\mathbb{S}^2$) For any conformal automorphism ψ of $\mathbb{S}^2$, the following holds:*

$$\langle\prod_{i=1}^N e^{\alpha_i\phi(\psi(z_i))}\rangle_{\mathbb{S}^2,\hat{g}} = \prod_{i=1}^N|\psi'(z_i)|^{-2\Delta_{\alpha_i}}\langle\prod_{i=1}^N e^{\alpha_i\phi(z_i)}\rangle_{\mathbb{S}^2,\hat{g}}. \tag{1.4.3}$$

The $\Delta_{\alpha_i} := \frac{\alpha_i}{2}(Q - \frac{\alpha_i}{2})$ in the above theorem are the so-called conformal weights of CFT. Similarly for a domain with boundary such as the unit disk $\mathbb{D}$ or the annulus Ω, see [40, 69]:

Theorem 1.4.4. *(KPZ formula on $\mathbb{D}$) For any conformal automorphism ψ of $\mathbb{D}$, the following holds:*

$$\langle\prod_{i=1}^N e^{\alpha_i\phi(\psi(z_i))}\prod_{j=1}^M e^{\frac{\beta_j}{2}\phi(\psi(s_j))}\rangle_{\mathbb{D}} = \prod_{i=1}^N|\psi'(z_i)|^{-2\Delta_{\alpha_i}}\prod_{j=1}^M|\psi'(s_j)|^{-\Delta_{\beta_j}}\langle\prod_{i=1}^N e^{\alpha_i\phi(z_i)}\prod_{j=1}^M e^{\frac{\beta_j}{2}\phi(s_j)}\rangle_{\mathbb{D}}. \tag{1.4.4}$$

Another very similar result that we have at our disposal is the conformal change of domain formula. This tells us how the theory behaves when we change domains by a conformal map ψ (for instance we can map the unit disk $\mathbb{D}$ to upper half plane $\mathbb{H}$). The result is again that the theory behaves as a conformal tensor.

Theorem 1.4.5. *(Conformal change of domain) Let D be a domain of $\mathbb{C}$ with a smooth boundary and conformally equivalent to the unit disk $\mathbb{D}$. Let $\psi : D \mapsto \mathbb{D}$ be a conformal map between D and $\mathbb{D}$ and let $g_\psi = |\psi'|^2 g(\psi)$ be the pull-back of a metric g on $\mathbb{D}$ by ψ. Then we have:*

$$\langle \prod_{i=1}^N e^{\alpha_i \phi(z_i)} \prod_{j=1}^M e^{\frac{\beta_j}{2}\phi(s_j)} \rangle_{D,g_\psi} = \prod_{i=1}^N |\psi'(z_i)|^{2\Delta_{\alpha_i}} \prod_{j=1}^M |\psi'(s_j)|^{\Delta_{\beta_j}} \langle \prod_{i=1}^N e^{\alpha_i \phi(\psi(z_i))} \prod_{j=1}^M e^{\frac{\beta_j}{2}\phi(\psi(s_j))} \rangle_{\mathbb{D},g}.$$

1.4.2 Ward identities

In this section we will explain the Ward identities of LCFT. For the reader who is only interested in the exact computations on GMC and LCFT presented in section 1.5, this subsection can be skipped. Nonetheless it is very interesting to verify that our probabilistic construction of LCFT satisfies these identities as they could lead in principle to the construction of the representation of the Virasoro algebra for LCFT.

To state these identities we will need to introduce the stress-energy tensor T of our theory. The general way to introduce this tensor is to take a derivative of correlations of LCFT with respect to the background metric g. For these purposes we will not work with a diagonal metric $g(x)dx^2$ but we consider the most general case $g = \sum_{i,j=1}^2 g_{ij} dx^i dx^j$. As is standard in the physics literature we will use the notation g^{ij} for the inverse matrix of g_{ij}, meaning that we have $\sum_{j=1}^2 g^{ij} g_{jk} = \delta^i_k$. To compute the derivative with respect to g we introduce an infinitesimal variation of g^{-1}, $g_\epsilon^{ij} := g^{ij} + \epsilon f^{ij}$ for some smooth f. In the same basis of coordinates the stress tensor T_{ij} is then defined heuristically by the following relation:

$$\frac{d}{d\epsilon}|_{\epsilon=0} \langle \prod_{i=1}^N e^{\alpha_i \phi(z_i)} \rangle_{\mathbb{S}^2, g_\epsilon} = \sum_{i,j=1}^2 \frac{1}{4\pi} \int_{\mathbb{C}} f^{ij}(z) \langle T_{ij}(z) \prod_{i=1}^N e^{\alpha_i \phi(z_i)} \rangle_{\mathbb{S}^2, g} d\lambda_g(z). \tag{1.4.5}$$

Here $d\lambda_g(z)$ stands for the volume form with respect to the metric g. Let us insist on the fact that the above definition is formal: it is the standard definition of the stress tensor that can be found in the physics literature. To turn this heuristic definition into a theorem of probability one would need to show that it is possible to exchange the derivative with respect to the background metric and the limit of the regularization procedure used to define the correlations. This is probably not very difficult but we leave it as an open problem. In any case, by a formal computation one can extract from (1.4.5) the expression of T . As is standard in the study of CFT it is more natural to express in terms of complex coordinates $z, \overline{z}$. The stress tensor then has only two non-trivial components, $T(z) := T_{zz}(z)$ and the anti-holomorphic counterpart $\overline{T}(z) := T_{\overline{z}\overline{z}}(z)$. One can show from (1.4.5) that formally:

$$T(z) = Q \partial_{zz}^2 \phi(z) - (\partial_z \phi(z))^2. \tag{1.4.6}$$

The first Ward identity claims the following,

$$\langle T(z) \prod_{i=1}^N e^{\alpha_i \phi(z_i)} \rangle_{\mathbb{S}^2, \hat{g}} = \sum_{i=1}^N \frac{\Delta_{\alpha_i}}{(z-z_i)^2} \langle T(z) \prod_{i=1}^N e^{\alpha_i \phi(z_i)} \rangle_{\mathbb{S}^2, \hat{g}} + \sum_{i=1}^N \frac{1}{z-z_i} \partial_{z_i} \langle \prod_{i=1}^N e^{\alpha_i \phi(z_i)} \rangle_{\mathbb{S}^2, \hat{g}}, \tag{1.4.7}$$

while the second Ward controls the singularity when two T-insertions come close:

$$\langle T(z)T(z')\prod_{i=1}^{N} e^{\alpha_i\phi(z_i)}\rangle_{\mathbb{S}^2,\hat{g}} = \frac{1}{2}\frac{c_L}{(z-z')^4}\langle\prod_{i=1}^{N} e^{\alpha_i\phi(z_i)}\rangle_{\mathbb{S}^2,\hat{g}} + \frac{2}{(z-z')^2}\langle T(z')\prod_{i=1}^{N} e^{\alpha_i\phi(z_i)}\rangle_{\mathbb{S}^2,\hat{g}}. + \frac{1}{z-z'}\partial_{z'}\langle T(z')\prod_{i=1}^{N} e^{\alpha_i\phi(z_i)}\rangle_{\mathbb{S}^2,\hat{g}} + \dots \tag{1.4.8}$$

Here the dots stand for a term that remains bounded as $z \to z'$. Both of these identities are shown in [17]. What is interesting about this second Ward identity is that we see the central charge of the theory appear, just like in Theorem 1.4.1 of the previous subsection.

So far we have only given an account of this story for LCFT on the Riemann sphere $\mathbb{S}^2$ but we can also write the first Ward identity for LCFT on a domain with boundary. For a correlation of LCFT on $\mathbb{H}$ we have:

$$\langle T(z)\prod_{i=1}^{N} e^{\alpha_i\phi(z_i)}\prod_{j=1}^{M} e^{\frac{\beta_j}{2}\phi(s_j)}\rangle_{\mathbb{H}} = (\sum_{i=1}^{N}\frac{\Delta_{\alpha_i}}{(z-z_i)^2} + \sum_{i=1}^{N}\frac{\Delta_{\alpha_i}}{(z-\overline{z_i})^2} + \sum_{j=1}^{M}\frac{\Delta_{\beta_j}}{(z-s_j)^2} + \sum_{i=1}^{N}\frac{1}{z-z_i}\partial_{z_i} + \sum_{i=1}^{N}\frac{1}{z-\overline{z_i}}\partial_{\overline{z_i}} + \sum_{j=1}^{M}\frac{1}{z-s_j}\partial_{s_j})\langle\prod_{i=1}^{N} e^{\alpha_i\phi(z_i)}\prod_{j=1}^{M} e^{\frac{\beta_j}{2}\phi(s_j)}\rangle_{\mathbb{H}}$$

Similarly as for the case of $\mathbb{S}^2$ a second Ward identity is also expected where one can see the central charge c_L appear.

1.4.3 Degenerate fields and BPZ equations

In this last subsection we explain the BPZ equations of Belavin-Polyakov-Zamolodchikov that will give us the constraints we need to compute certain correlation functions of LCFT. We start by explaining the case of the Riemann sphere $\mathbb{S}^2$. Consider the following function,

$$z \to \langle e^{\chi\phi(z)}\prod_{i=1}^{N} e^{\alpha_i\phi(z_i)}\rangle_{\mathbb{S}^2,\hat{g}}, \tag{1.4.9}$$

where the points $z, z_i \in \mathbb{S}^2$ have respective weights χ, α_i chosen so that the Seiberg bounds (1.3.6) hold. The reason why we distinguish the point z of weight χ is that we will choose this χ to be equal to a very special value, either $-\frac{\gamma}{2}$ or $-\frac{2}{\gamma}$. $e^{\chi\phi(z)}$ is then called a degenerate field insertion and the value of χ is the degenerate weight. Under this very specific condition the function of z given by (1.4.9) will be solution to a second order differential equation. Indeed, it is shown in [46] using probabilistic techniques:

Theorem 1.4.6. *Consider points $z, z_i \in \mathbb{S}^2$ with respective weights $\chi, \alpha_i \in \mathbb{R}$ chosen so that the Seiberg bounds (1.3.6) hold. Then we have the following PDE,*

$$(\frac{1}{\chi^2}\partial_{zz} + \sum_{i=1}^{N}\frac{\Delta_{\alpha_i}}{(z-z_i)^2} + \sum_{i=1}^{N}\frac{1}{z-z_i}\partial_{z_i})\langle e^{\chi\phi(z)}\prod_{i=1}^{N} e^{\alpha_i\phi(z_i)}\rangle_{\mathbb{S}^2,\hat{g}} = 0,$$

under the condition that χ is worth $-\frac{\gamma}{2}$ or $-\frac{2}{\gamma}$.

Again we see the conformal weight $\Delta_{\alpha_i} := \frac{\alpha_i}{2}(Q - \frac{\alpha_i}{2})$ appear. For the purposes of proving the DOZZ formula the authors of [47] have applied the differential equation more specifically to the case of a four-point function, see subsection 1.5.1. Naturally a similar story can be told for the case of a domain with boundary. It will be more convenient to write the differential equation on $\mathbb{H}$ although by conformal mapping we can easily transform it into an equation on $\mathbb{D}$. For this boundary case we consider correlations of the type $\langle \prod_{i=1}^N e^{\alpha_i \phi(z_i)} \prod_{j=1}^M e^{\frac{\beta_j}{2}\phi(s_j)} \rangle_{\mathbb{H}}$ where we have bulk insertions $z_i \in \mathbb{H}$ with $\alpha_i \in \mathbb{R}$ and boundary insertions $s_j \in \partial\mathbb{H} = \mathbb{R}$ with $\beta_j \in \mathbb{R}$ and again we choose the weights so that the bounds (1.3.21) hold. Now in the boundary case there are two possibilities for the position of the degenerate insertion, either in the bulk $\mathbb{H}$ or on the boundary $\mathbb{R}$. Both of these cases will lead to BPZ equations. We start by giving the theorem corresponding to the case where the degenerate insertion is placed in the bulk:

Theorem 1.4.7. *Consider points $z, z_i \in \mathbb{H}$ with respective weights χ, $\alpha_i \in \mathbb{R}$ and points $s_j \in \mathbb{R}$ with respective weights $\beta_j \in \mathbb{R}$ chosen so that the Seiberg bounds (1.3.21) hold. Then we have the following PDE,*

$$\Big(\frac{1}{\chi^2}\partial_{zz} + \frac{\Delta_\chi}{(z-\overline{z})^2} + \sum_{i=1}^N \frac{\Delta_{\alpha_i}}{(z-z_i)^2} + \sum_{i=1}^N \frac{\Delta_{\alpha_i}}{(z-\overline{z_i})^2} + \sum_{j=1}^M \frac{\Delta_{\beta_j}}{(z-s_j)^2}$$
$$+ \frac{1}{z-\overline{z}}\partial_{\overline{z}} + \sum_{i=1}^N \frac{1}{z-z_i}\partial_{z_i} + \sum_{i=1}^N \frac{1}{z-\overline{z_i}}\partial_{\overline{z_i}} + \sum_{j=1}^M \frac{1}{z-s_j}\partial_{s_j}\Big)\langle e^{\chi\phi(z)} \prod_{i=1}^N e^{\alpha_i\phi(z_i)} \prod_{j=1}^M e^{\frac{\beta_j}{2}\phi(s_i)}\rangle_{\mathbb{H}} = 0,$$

where χ needs to be equal to $-\frac{\gamma}{2}$ or $-\frac{2}{\gamma}$.

In chapter 2 we only prove this theorem in the very special case where $\mu = 0$ and with two insertion points, the degenerate insertion z of weight $-\frac{\gamma}{2}$ plus another point z_1 of weight α. We restrict ourselves to this simple case as this is all that is required to prove the Fyodorov-Bouchaud formula. Moving on to the second possibility where we place the degenerate insertion on the boundary, we also expect to have a BPZ equation of order 2 but it turns out that for such an equation to hold we must choose $\mu = 0$ in our Theorem 1.3.2 that defines the correlations on $\mathbb{D}$.

Theorem 1.4.8. *Suppose $\mu = 0$ and $\mu_\partial \neq 0$. Consider points $z_i \in \mathbb{H}$ with weights $\alpha_i \in \mathbb{R}$ and points $s, s_j \in \mathbb{R}$ with respective weights $\chi, \beta_j \in \mathbb{R}$ chosen so that the Seiberg bounds (1.3.21) hold. Then we have the following PDE,*

$$\Big(\frac{1}{\chi^2}\partial_{ss} + \sum_{i=1}^N \frac{\Delta_{\alpha_i}}{(s-z_i)^2} + \sum_{i=1}^N \frac{\Delta_{\alpha_i}}{(s-\overline{z}_i)^2} + \sum_{j=1}^M \frac{\Delta_{\beta_j}}{(s-s_j)^2}$$
$$+ \sum_{i=1}^N \frac{1}{s-z_i}\partial_{z_i} + \sum_{i=1}^N \frac{1}{s-\overline{z}_i}\partial_{\overline{z}_i} + \sum_{j=1}^M \frac{1}{s-s_j}\partial_{s_j}\Big)\langle e^{\frac{\chi}{2}\phi(s)} \prod_{i=1}^N e^{\alpha_i\phi(z_i)} \prod_{j=1}^M e^{\frac{\beta_j}{2}\phi(s_i)}\rangle_{\mathbb{H}} = 0,$$

where χ is worth $-\frac{\gamma}{2}$ or $-\frac{2}{\gamma}$.

One may wonder if it is nonetheless possible to write a BPZ equation with a boundary degenerate insertion in the case where $\mu, \mu_\partial > 0$. It should be possible that an equation of order 3 holds although this still requires to be checked. In fact let us mention that CFT predicts there to be BPZ equations of all order, not only of order 2 or 3.

1.5 Integrability of GMC and of Liouville theory

1.5.1 The DOZZ formula on the Riemann sphere

In this section we will sketch the proof of the celebrated DOZZ formula which gives the value of the three-point correlation function of LCFT on $\mathbb{S}^2$. In [47] the choice of background metric is $\tilde{g}(x) = |x|_+^{-4}$ where $|x|_+ = \max(|x|, 1)$ but of course thanks to Theorem 1.4.1 this choice is irrelevant. We state the main result of [47]:

Theorem 1.5.1. *(DOZZ formula) Choose three points $z_1, z_2, z_3 \in \mathbb{S}^2$ of respective weights $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that the bounds (1.3.6) are satisfied. Then we have the following expression for the three-point correlation function,*

$$\langle \prod_{i=1}^{3} e^{\alpha_i \phi(z_i)} \rangle_{\mathbb{S}^2, \tilde{g}} = |z_1 - z_2|^{2\Delta_{12}} |z_2 - z_3|^{2\Delta_{23}} |z_1 - z_3|^{2\Delta_{13}} C_\gamma(\alpha_1, \alpha_2, \alpha_3), \tag{1.5.1}$$

where we have:

$$C_\gamma(\alpha_1, \alpha_2, \alpha_3) = (\pi \mu l(\frac{\gamma^2}{4})(\frac{\gamma}{2})^{2-\frac{\gamma^2}{2}})^{\frac{2Q-\overline{\alpha}}{\gamma}} \frac{\Upsilon'_{\frac{\gamma}{2}}(0)\Upsilon_{\frac{\gamma}{2}}(\alpha_1)\Upsilon_{\frac{\gamma}{2}}(\alpha_2)\Upsilon_{\frac{\gamma}{2}}(\alpha_3)}{\Upsilon_{\frac{\gamma}{2}}(\frac{\overline{\alpha}}{2} - Q)\Upsilon_{\frac{\gamma}{2}}(\frac{\overline{\alpha}}{2} - \alpha_1)\Upsilon_{\frac{\gamma}{2}}(\frac{\overline{\alpha}}{2} - \alpha_2)\Upsilon_{\frac{\gamma}{2}}(\frac{\overline{\alpha}}{2} - \alpha_3)}. \tag{1.5.2}$$

In the above expressions we have $l(x) = \frac{\Gamma(x)}{\Gamma(1-x)}$, $\overline{\alpha} = \alpha_1 + \alpha_2 + \alpha_3$, $\Delta_{12} = \Delta_{\alpha_3} - \Delta_{\alpha_2} - \Delta_{\alpha_1}$, $\Delta_{\alpha_i} = \frac{\alpha_i}{2}(Q - \frac{\alpha_i}{2})$, $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$ and Υ_γ is a special function expressed in terms of the double gamma function Γ_γ,

$$\Upsilon_\gamma(x) = \frac{1}{\Gamma_\gamma(x)\Gamma_\gamma(Q-x)}, \tag{1.5.3}$$

where:

$$\ln \Gamma_\gamma(x) = \int_0^\infty \frac{dt}{t} \left[\frac{e^{-xt} - e^{-\frac{Qt}{2}}}{(1 - e^{-\frac{\gamma t}{2}})(1 - e^{-\frac{2t}{\gamma}})} - \frac{(\frac{Q}{2} - x)^2}{2} e^{-t} + \frac{x - \frac{Q}{2}}{t} \right]. \tag{1.5.4}$$

We will now briefly mention the main arguments leading to the DOZZ formula. The first part of the theorem that gives the dependence on the positions z_1, z_2, z_3 is a consequence of the KPZ formula (1.4.3). Indeed one simply needs to apply (1.4.3) to the Mobius map ψ that satisfies $\psi(z_1) = 0$, $\psi(z_2) = 1$, $\psi(z_3) = \infty$. The structure constant $C_\gamma(\alpha_1, \alpha_2, \alpha_3)$ is then recovered by taking the following limit:

$$C_\gamma(\alpha_1, \alpha_2, \alpha_3) = \lim_{z_3 \to \infty} |z_3|^{4\Delta_3} \langle e^{\alpha_1 \phi(0)} e^{\alpha_2 \phi(1)} e^{\alpha_3 \phi(z_3)} \rangle_{\mathbb{S}^2, \tilde{g}}. \tag{1.5.5}$$

The difficult part of the theorem is thus to give the exact value for the constant $C_\gamma(\alpha_1, \alpha_2, \alpha_3)$. This is where we need to use the two BPZ equations of Theorem 1.4.6 for a four-point function. There are two such equations as the degenerate weight can be equal either to $-\frac{\gamma}{2}$ or to $-\frac{2}{\gamma}$. One can then deduce two non-trivial relations on $C_\gamma(\alpha_1, \alpha_2, \alpha_3)$ which are,

$$\frac{C_\gamma(\alpha_1 + \frac{\gamma}{2}, \alpha_2, \alpha_3)}{C_\gamma(\alpha_1 - \frac{\gamma}{2}, \alpha_2, \alpha_3)} = -\frac{1}{\pi \mu} \frac{l(-\frac{\gamma^2}{4}) l(\frac{\gamma \alpha_1}{2}) l(\frac{\alpha_1 \gamma}{2} - \frac{\gamma^2}{4}) l(\frac{\gamma}{4}(\bar{\alpha} - 2\alpha_1 - \frac{\gamma}{2}))}{l(\frac{\gamma}{4}(\bar{\alpha} - \frac{\gamma}{2} - 2Q)) l(\frac{\gamma}{4}(\bar{\alpha} - 2\alpha_3 - \frac{\gamma}{2})) l(\frac{\gamma}{4}(\bar{\alpha} - 2\alpha_2 - \frac{\gamma}{2}))}, \tag{1.5.6}$$

and

$$\frac{C_\gamma(\alpha_1 + \frac{2}{\gamma}, \alpha_2, \alpha_3)}{C_\gamma(\alpha_1 - \frac{2}{\gamma}, \alpha_2, \alpha_3)} = -\frac{1}{\pi \tilde{\mu}} \frac{l(-\frac{4}{\gamma^2}) l(\frac{2\alpha_1}{\gamma}) l(\frac{2\alpha_1}{\gamma} - \frac{4}{\gamma^2}) l(\frac{1}{\gamma}(\bar{\alpha} - 2\alpha_1 - \frac{2}{\gamma}))}{l(\frac{1}{\gamma}(\bar{\alpha} - \frac{2}{\gamma} - 2Q)) l(\frac{1}{\gamma}(\bar{\alpha} - 2\alpha_3 - \frac{2}{\gamma})) l(\frac{1}{\gamma}(\bar{\alpha} - 2\alpha_2 - \frac{2}{\gamma}))}, \tag{1.5.7}$$

with $\tilde{\mu} = \frac{(\mu\pi l(\frac{\gamma^2}{2}))^{\frac{4}{\gamma^2}}}{\pi l(\frac{4}{\gamma^2})}$ being the dual cosmological constant. Of course by symmetry similar relations are verified for a shift on α_2 or on α_3. It turns out that these two shift equations completely determine the dependence on the α_i if $\gamma^2 \notin \mathbb{Q}$. Then one can extent the formula to all values of γ by a standard continuity argument. For a much more complete summary of the proof of the DOZZ formula one can take a look at the lecture notes [82].

1.5.2 The Fyodorov-Bouchaud formula, *Summary of chapter 2*

The second integrability result explained in detail in chapter 2 is a proof of the Fyodorov-Bouchaud formula. Consider the log-correlated field X on the unit circle $\partial\mathbb{D}$ with covariance:

$$\mathbb{E}[X(e^{i\theta})X(e^{i\theta'})] = 2\ln\frac{1}{|e^{i\theta} - e^{i\theta'}|}. \tag{1.5.8}$$

We will prove in chapter 2 the following theorem:

Theorem 1.5.2. *(Fyodorov-Bouchaud formula) Let $\gamma \in (0,2)$. For all real*[6] *p such that $p < \frac{4}{\gamma^2}$ the following identity holds:*

$$\mathbb{E}[(\frac{1}{2\pi}\int_0^{2\pi} e^{\frac{\gamma}{2}X(e^{i\theta})}d\theta)^p] = \frac{\Gamma(1-p\frac{\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})^p}. \tag{1.5.9}$$

As a consequence, the variable $\frac{1}{2\pi}\int_0^{2\pi} e^{\frac{\gamma}{2}X(e^{i\theta})}d\theta$ has an explicit density given by,

$$f(y) = \frac{4\beta}{\gamma^2}(\beta y)^{-\frac{4}{\gamma^2}-1}e^{-(\beta y)^{-\frac{4}{\gamma^2}}}\mathbf{1}_{[0,\infty[}(y), \tag{1.5.10}$$

where we have set $\beta = \Gamma(1-\frac{\gamma^2}{4})$. Finally a third equivalent way of stating the result is by the equality in law,

$$\frac{1}{2\pi}\int_0^{2\pi} e^{\frac{\gamma}{2}X(e^{i\theta})}d\theta \overset{law}{=} \frac{1}{\beta}Z^{-\frac{\gamma^2}{4}}, \tag{1.5.11}$$

where Z is an exponential random variable of parameter 1.

What is conjectured in the paper [31] by Fyodorov and Bouchaud is the value for the moments (1.5.9) but of course (1.5.10) and (1.5.11) give completely equivalent statements of the theorem. We will now sketch the main arguments of the proof. The clever observation of [31] that allows to conjecture the result (1.5.9) is that it is possible to compute the moments in the very special case where $p \in \mathbb{N}$ with $p < \frac{4}{\gamma^2}$:

[6] It is easy to see that the result also holds for all complex p such that $\mathrm{Re}(p) < \frac{4}{\gamma^2}$.

$$
\begin{aligned}
\mathbb{E}[(\frac{1}{2\pi}\int_0^{2\pi} e^{\frac{\gamma}{2}X(e^{i\theta})}d\theta)^p] &= \lim_{\epsilon\to 0}\mathbb{E}[(\frac{1}{2\pi}\int_0^{2\pi} e^{\frac{\gamma}{2}X_\epsilon(e^{i\theta})-\frac{\gamma^2}{8}\mathbb{E}[X_\epsilon(e^{i\theta})^2]}d\theta)^p] \\
&= \lim_{\epsilon\to 0}\frac{1}{(2\pi)^p}\int_{[0,2\pi]^p}\mathbb{E}[\prod_{i=1}^p e^{\frac{\gamma}{2}X_\epsilon(e^{i\theta_i})-\frac{\gamma^2}{8}\mathbb{E}[X_\epsilon(e^{i\theta_i})^2]}]d\theta_1\dots d\theta_p \\
&= \frac{1}{(2\pi)^p}\int_{[0,2\pi]^p} e^{\frac{\gamma^2}{4}\sum_{i<j}\mathbb{E}[X(e^{i\theta_i})X(e^{i\theta_j})]}d\theta_1\dots d\theta_p \\
&= \frac{1}{(2\pi)^p}\int_{[0,2\pi]^p}\prod_{i<j}\frac{1}{|e^{i\theta_i}-e^{i\theta_j}|^{\frac{\gamma^2}{2}}}d\theta_1\dots d\theta_p \\
&= \frac{\Gamma(1-\frac{p\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})}.
\end{aligned}
$$

Notice that the reason why this computations works is that we can write p integrals and exchange them with the expectation value $\mathbb{E}[\cdot]$. We land on a real integral over p angles θ_i whose value is explicit known as a ratio of Gamma functions. This is a circular variant of the famous Selberg integral. Now it is tempting to want to replace the $p \in \mathbb{N}$ by a $p \in \mathbb{R}$ smaller then $\frac{4}{\gamma^2}$. But of course it is not because two functions coincide on a finite number of points that they coincide on their domains of definition.

To overcome this major difficulty and prove Theorem 1.5.2 we will use again the techniques of CFT, but this time for LCFT of the unit disk $\mathbb{D}$ (or equivalently on the upper half-plane $\mathbb{H}$). The key observation is to realize that the inverse moments of GMC integrated on the unit circle $\partial\mathbb{D}$ can be expressed as one-point correlation functions of LCFT on the unit disk in the case where we chose $\mu = 0$, $\mu_\partial > 0$, see equation (1.3.23) of section 1.3.2. We can then use the BPZ equations of Theorem 1.4.7 to extract non-trivial relations on these moments. Introducing the notations of chapter 2, let p be a real number such that $p < \frac{4}{\gamma^2}$ and let:

$$
U(\gamma,p) = \mathbb{E}[(\int_0^{2\pi} e^{\frac{\gamma}{2}X(e^{i\theta})}d\theta)^p]. \tag{1.5.12}
$$

The proof is based on studying the following function or "observable" defined for $t \in [0,1]$:

$$
G(\gamma,p,t) = \mathbb{E}[(\int_0^{2\pi}|t-e^{i\theta}|^{\frac{\gamma^2}{2}}e^{\frac{\gamma}{2}X(e^{i\theta})}d\theta)^p]. \tag{1.5.13}
$$

The reason why this is the right auxiliary function to look at is that the two-point correlation function of LCFT on $\mathbb{D}$ can be expressed simply in terms of $G(\gamma,p,t)$, see (1.3.23),

$$
\begin{aligned}
\langle e^{-\frac{\gamma}{2}\phi(t)}e^{\alpha\phi(0)}\rangle_{\mathbb{D}} &= \frac{2}{\gamma}\mu_\partial^{-\frac{2\alpha-\gamma-2Q}{\gamma}}\Gamma(\frac{2\alpha-\gamma-2Q}{\gamma})t^{\frac{\alpha\gamma}{2}}(1-t^2)^{-\frac{\gamma^2}{8}}\mathbb{E}\left[(\int_{\partial\mathbb{D}} e^{\frac{\gamma}{2}(X(e^{i\theta})-2\alpha\ln|e^{i\theta}|+\gamma\ln|e^{i\theta}-t|)}d\theta)^{-\frac{2\alpha-\gamma-2Q}{\gamma}}\right] \\
&= \frac{2}{\gamma}\mu_\partial^{-\frac{2\alpha-\gamma-2Q}{\gamma}}\Gamma(\frac{2\alpha-\gamma-2Q}{\gamma})t^{\frac{\alpha\gamma}{2}}(1-t^2)^{-\frac{\gamma^2}{8}}G(\gamma,p,t),
\end{aligned} \tag{1.5.14}
$$

where the link between α and p is given by $\gamma p = 2Q + \gamma - 2\alpha$. Furthermore the KPZ relation of Theorem 1.4.4 combined with the change of domain formula of Theorem 1.4.5 will give us the relation:

$$
\langle e^{-\frac{\gamma}{2}\phi(z)}e^{\alpha\phi(z_1)}\rangle_{\mathbb{H}} = \frac{1}{|z_1-\overline{z}_1|^{2\Delta_\alpha-2\Delta_{-\frac{\gamma}{2}}}}\frac{1}{|z-\overline{z}_1|^{4\Delta_{-\frac{\gamma}{2}}}}\langle e^{-\frac{\gamma}{2}\phi(t)}e^{\alpha\phi(0)}\rangle_{\mathbb{D}}. \tag{1.5.15}
$$

Now starting from the BPZ equation of Theorem 1.4.7 for the function $z \to \langle e^{-\frac{\gamma}{2}\phi(z)} e^{\alpha_1 \phi(z_1)} \rangle_{\mathbb{H}}$,

$$(\frac{4}{\gamma^2}\partial_{zz} + \frac{\Delta_{-\frac{\gamma}{2}}}{(z-\overline{z})^2} + \frac{\Delta_\alpha}{(z-z_1)^2} + \frac{\Delta_\alpha}{(z-\overline{z_1})^2} + \frac{1}{z-\overline{z}}\partial_{\overline{z}} + \frac{1}{z-z_1}\partial_{z_1} + \frac{1}{z-\overline{z_1}}\partial_{\overline{z_1}})\langle e^{-\frac{\gamma}{2}\phi(z)} e^{\alpha_1\phi(z_1)}\rangle_{\mathbb{H}} = 0, \tag{1.5.16}$$

by a long but straightforward change of variable one lands on an equation for the function $G(\gamma, p, t)$:

Proposition 1.5.3. *(BPZ) For $\gamma \in (0,2)$ and $p < 0$ the function $t \mapsto G(\gamma, p, t)$ satisfies the following differential equation,*

$$(t(1-t^2)\frac{\partial^2}{\partial t^2} + (t^2-1)\frac{\partial}{\partial t} + 2(C - (A+B+1)t^2)\frac{\partial}{\partial t} - 4ABt)G(\gamma, p, t) = 0,$$

with the following values for A, B, and C:

$$A = -\frac{\gamma^2 p}{4}, \;\; B = -\frac{\gamma^2}{4}, \;\; C = \frac{\gamma^2}{4}(1-p) + 1.$$

This equation may look convoluted but it turns out we can write its space of solutions using the theory of hypergeometric equations. The solution space is a two-dimensional vector space and we will write its solutions in two different bases, one that corresponds to an expansion in powers of t and one to an expansion in power of $1-t$:

$$G(\gamma,p,t) = C_1 F(-\frac{\gamma^2 p}{4}, -\frac{\gamma^2}{4}, \frac{\gamma^2}{4}(1-p)+1, t^2) + C_2 t^{\frac{\gamma^2}{2}(p-1)} F(-\frac{\gamma^2}{4}, \frac{\gamma^2}{4}(p-2), \frac{\gamma^2}{4}(p-1)+1, t^2)$$

and

$$G(\gamma,p,t) = B_1 F(-\frac{\gamma^2 p}{4}, -\frac{\gamma^2}{4}, -\frac{\gamma^2}{2}, 1-t^2) + B_2 (1-t^2)^{1+\frac{\gamma^2}{2}} F(1+\frac{\gamma^2}{4}, \frac{\gamma^2}{4}(2-p)+1, 2+\frac{\gamma^2}{2}, 1-t^2)$$

where F is the standard hypergeometric series. The coefficients C_1, C_2 and B_1, B_2 are real constants that depend on γ and p and they are linked by an explicit change of basis formula given by the theory of hypergeometric equations.

The last part of the proof is based on exploiting the fact that the above coefficients C_1, C_2, B_1 and B_2 can be identified in terms of $U(\gamma, p)$ by performing asymptotic expansions directly on the expression of $G(\gamma, p, t)$. Notice for instance that $C_1 = G(\gamma, p, 0) = U(\gamma, p)$. We also express B_2 in terms of $U(\gamma, p-1)$ and find $C_2 = 0$. The change of basis then leads to:

Proposition 1.5.4. *For all $\gamma \in (0,2)$ and for $p \leq 0$, we have the relation:*

$$U(\gamma, p) = \frac{2\pi \Gamma(1 - p\frac{\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})\Gamma(1-(p-1)\frac{\gamma^2}{4})} U(\gamma, p-1).$$

From this shift equation we deduce recursively all the negative moments of the variable $\frac{1}{2\pi}\int_0^{2\pi} e^{\frac{\gamma}{2}X(e^{i\theta})} d\theta$, i.e. we get:

$$\mathbb{E}[(\frac{1}{2\pi}\int_0^{2\pi} e^{\frac{\gamma}{2}X(e^{i\theta})} d\theta)^{-n}] = \Gamma(1+\frac{n\gamma^2}{4})\Gamma(1-\frac{\gamma^2}{4})^n, \quad \forall n \in \mathbb{N}. \tag{1.5.17}$$

The series

$$\lambda \mapsto \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\Gamma(1+\frac{n\gamma^2}{4})\Gamma(1-\frac{\gamma^2}{4})^n$$

has an infinite radius of convergence, meaning that these negative moments determine the law of $\frac{1}{2\pi}\int_0^{2\pi} e^{\frac{\gamma}{2}X(e^{i\theta})} d\theta$ and since we can find a probability density (1.5.10), this completes the proof of Theorem 1.5.2.

1.5.3 GMC on the unit interval, *Summary of chapter 3*

Lastly we summarize here a third exact computation presented in detail in chapter 3. We study this time the GMC measure associated to the log-correlated field X_I defined on the unit interval $[0,1]$. Again we will prove an exact formula for the moments of the total mass of the GMC measure. What is remarkable in the case of the unit interval is that similar techniques as the ones presented for the case of the unit circle also work although there is a priori no CFT behind our model. The log-correlated field X_I that we work with has a covariance given by:

$$\mathbb{E}[X_I(x)X_I(y)] = 2\ln\frac{1}{|x-y|}. \tag{1.5.18}$$

For this model of GMC the most general quantity that we will provide an expression for is defined for $\gamma \in (0,2)$ and for real p, a, b obeying the bounds (1.5.20) written below:

$$M(\gamma,p,a,b) := \mathbb{E}[(\int_0^1 x^a(1-x)^b e^{\frac{\gamma}{2}X_I(x)}dx)^p]. \tag{1.5.19}$$

This is the moment p of the total mass of our GMC measure with two "insertion points" in 0 and 1 of weight a and b. As explained in subsection 1.2.3, the theory of Gaussian multiplicative chaos tells us that these moments are non-trivial, i.e. different from 0 and $+\infty$, if and only if:

$$a > -\frac{\gamma^2}{4} - 1, \quad b > -\frac{\gamma^2}{4} - 1, \quad p < \frac{4}{\gamma^2} \wedge (1 + \frac{4}{\gamma^2}(1+a)) \wedge (1 + \frac{4}{\gamma^2}(1+b)). \tag{1.5.20}$$

We state here the main result of chapter 3:

Theorem 1.5.5. *For $\gamma \in (0,2)$ and for p, a, b satisfying (1.5.20)[7], $M(\gamma,p,a,b)$ is given by,*

$$\frac{(2\pi)^p(\frac{2}{\gamma})^{p\frac{\gamma^2}{4}}}{\Gamma(1-\frac{\gamma^2}{4})^p}\frac{\Gamma_\gamma(\frac{2}{\gamma}(a+1)-(p-1)\frac{\gamma}{2})\Gamma_\gamma(\frac{2}{\gamma}(b+1)-(p-1)\frac{\gamma}{2})\Gamma_\gamma(\frac{2}{\gamma}(a+b+2)-(p-2)\frac{\gamma}{2})\Gamma_\gamma(\frac{2}{\gamma}-p\frac{\gamma}{2})}{\Gamma_\gamma(\frac{2}{\gamma})\Gamma_\gamma(\frac{2}{\gamma}(a+1)+\frac{\gamma}{2})\Gamma_\gamma(\frac{2}{\gamma}(b+1)+\frac{\gamma}{2})\Gamma_\gamma(\frac{2}{\gamma}(a+b+2)-(2p-2)\frac{\gamma}{2})},$$

where the function $\Gamma_\gamma(x)$ is defined by equation (1.5.4).

As a corollary by choosing $a=b=0$ we obtain the value of the moments of the GMC measure without insertions:

Corollary 1.5.6. *For $\gamma \in (0,2)$ and $p < \frac{4}{\gamma^2}$:*

$$\mathbb{E}[(\int_0^1 e^{\frac{\gamma}{2}X_I(x)}dx)^p] = \frac{(2\pi)^p(\frac{2}{\gamma})^{p\frac{\gamma^2}{4}}}{\Gamma(1-\frac{\gamma^2}{4})^p}\frac{\Gamma_\gamma(\frac{2}{\gamma}-(p-1)\frac{\gamma}{2})^2\Gamma_\gamma(\frac{4}{\gamma}-(p-2)\frac{\gamma}{2})\Gamma_\gamma(\frac{2}{\gamma}-p\frac{\gamma}{2})}{\Gamma_\gamma(\frac{2}{\gamma})\Gamma_\gamma(\frac{2}{\gamma}+\frac{\gamma}{2})^2\Gamma_\gamma(\frac{4}{\gamma}-(2p-2)\frac{\gamma}{2})}. \tag{1.5.21}$$

Theorem 1.5.5 was first conjectured in [60] and Corollary 1.5.6 was first conjectured in [35]. The idea of the proof of Theorem 1.5.5 is the same as the proof of the Fyodorov-Bouchaud formula. We introduce the following auxiliary functions,

$$U(t) := \mathbb{E}[(\int_0^1 (x-t)^{\frac{\gamma^2}{4}} x^a(1-x)^b e^{\frac{\gamma}{2}X_I(x)}dx)^p], \tag{1.5.22}$$

and

$$\tilde{U}(t) := \mathbb{E}[(\int_0^1 (x-t)\, x^a(1-x)^b\, e^{\frac{\gamma}{2}X_I(x)}dx)^p], \tag{1.5.23}$$

and show using probabilistic techniques that the following holds:

[7] Again the result also holds for all complex p such that $\mathrm{Re}(p)$ satisfies the bounds (1.5.20).

Proposition 1.5.7. *Let $0 < \gamma < 2$, a, b, p satisfy the bounds (1.5.20), then U is solution of the hypergeometric equation for $t < 0$:*

$$t(1-t)U''(t) + (C - (A+B+1)t)U'(t) - ABU(t) = 0. \tag{1.5.24}$$

The parameters are given by:

$$A = -\frac{p\gamma^2}{4}, B = -(a+b+1) - (2-p)\frac{\gamma^2}{4}, C = -a - \frac{\gamma^2}{4}. \tag{1.5.25}$$

$\tilde{U}$ is also solution of the hypergeometric equation with different parameters:

$$\tilde{A} = -p, \tilde{B} = -\frac{4}{\gamma^2}(a+b+2) + p - 1, \tilde{C} = -\frac{4}{\gamma^2}(a+1). \tag{1.5.26}$$

The reason why we need two differential equations to prove Theorem 1.5.5 is that because of the presence of the insertions a and b we need two shift equations to completely determine the dependence of (1.5.19) on a and b. Therefore the techniques of proof of this formula on $[0,1]$ are much closer in spirit to the techniques used to prove the DOZZ formula than the ones used to prove the Fyodorov-Bouchaud formula.

A mystery about this GMC on the interval is that it is not clear whether there exists an actual CFT where $U(t)$ and $\tilde{U}(t)$ correspond to correlations with degenerate insertions which would explain why the differential equations of Proposition 1.5.7 hold. Furthermore if we replace the real t by a complex variable $t \in \mathbb{C}\backslash[0,\infty]$, it is not hard to see that $U(t)$ is a holomorphic function and Proposition 1.5.7 will hold if we replace the ordinary derivative by a complex derivative ∂_t. In the conformal bootstrap approach of CFT initiated by Belavin-Polyakov-Zamolodchikov in [8], a correlation function with a degenerate insertion can be decomposed into combinations of the structure constants and conformal blocks. A conformal block is a locally holomorphic function and it is always accompanied by its complex conjugate in the decomposition. What is mysterious with $U(t)$ and $\tilde{U}(t)$ is that we only see the holomorphic part. At this stage we have no CFT-based explanation for this observation although a possible path could be to look at boundary LCFT with multiple boundary cosmological constants, see for instance [57].

Finally we state that Theorem 1.5.5 can be used to obtain the following tail expansion for a GMC measure in dimension one. In the result appears $\overline{R}_1^{\partial}(\alpha)$, a one-dimensional reflection coefficient.

Proposition 1.5.8. *For $\gamma \in (0,2)$ and $\eta \in (0,1)$ define:*

$$I_{1,\eta}^{\partial}(\alpha) := \int_0^{\eta} x^{-\frac{\gamma\alpha}{2}} e^{\frac{\gamma}{2}X_I(x)} dx. \tag{1.5.27}$$

Then for $\alpha \in (\frac{\gamma}{2}, Q)$ we have the following tail expansion for $I_{1,\eta}^{\partial}(\alpha)$ as $u \to \infty$ and for some $\nu > 0$,

$$\mathbb{P}(I_{1,\eta}^{\partial}(\alpha) > u) = \frac{\overline{R}_1^{\partial}(\alpha)}{u^{\frac{2}{\gamma}(Q-\alpha)}} + O(\frac{1}{u^{\frac{2}{\gamma}(Q-\alpha)+\nu}}), \tag{1.5.28}$$

where the value of $\overline{R}_1^{\partial}(\alpha)$ is given by:

$$\overline{R}_1^{\partial}(\alpha) = \frac{(2\pi)^{\frac{2}{\gamma}(Q-\alpha)-\frac{1}{2}}(\frac{2}{\gamma})^{\frac{\gamma}{2}(Q-\alpha)-\frac{1}{2}}}{(Q-\alpha)\Gamma(1-\frac{\gamma^2}{4})^{\frac{2}{\gamma}(Q-\alpha)}} \frac{\Gamma_\gamma(\alpha-\frac{\gamma}{2})}{\Gamma_\gamma(Q-\alpha)}. \tag{1.5.29}$$

1.6 Applications and perspectives

Finally, this section provides a list of applications of GMC and of Liouville theory to other problems in probability theory and in mathematical physics. We also provide perspectives on future research.

1.6.1 The maximum of X and random matrix theory

We now turn to the applications of our exact formulas for the GMC measures. It turns out that the Fyodorov-Bouchaud formula will give us some precise information on the behaviour of the maximum of our field X on the unit circle. Characterizing the behaviour of the maximum of X requires to compute the law of the total mass of the derivative martingale,

$$Y' := -\frac{1}{2}\int_0^{2\pi} X(e^{i\theta})e^{X(e^{i\theta})}d\theta, \tag{1.6.1}$$

which following [6] can be characterized by the convergence in law:

$$2Y' = \lim_{\gamma\to 2}\frac{1}{2-\gamma}Y_\gamma. \tag{1.6.2}$$

Therefore using Theorem 1.5.2 we can easily compute the density for $2Y'$:

$$f_{2Y'}(y) = y^{-2}e^{-y^{-1}}\mathbf{1}_{[0,\infty[}(y).$$

We observe that $\ln 2Y'$ is distributed like a standard Gumbel law. Recall that an impressive series of works (see [13, 15] for the latest results) have proven that for suitable sequences of cut-off approximations X_ϵ the following convergence in law holds,

$$\max_{\theta\in[0,2\pi]} X_\epsilon(e^{i\theta}) - 2\ln\frac{1}{\epsilon} + \frac{3}{2}\ln\ln\frac{1}{\epsilon} \underset{\epsilon\to 0}{\to} \mathcal{G} + \ln Y' + C, \tag{1.6.3}$$

where $\mathcal{G}$ is a Gumbel law independent from Y' and C is a non universal constant that depends on the cut-off procedure. From this convergence and previous considerations, one can deduce the following convergence in law,

$$\max_{\theta\in[0,2\pi]} X_\epsilon(e^{i\theta}) - 2\ln\frac{1}{\epsilon} + \frac{3}{2}\ln\ln\frac{1}{\epsilon} \underset{\epsilon\to 0}{\to} \mathcal{G}_1 + \mathcal{G}_2 + C, \tag{1.6.4}$$

where $\mathcal{G}_1$ and $\mathcal{G}_2$ are two independent Gumbel laws and where we have absorbed the factor $\ln 2$ in the constant C. This convergence was conjectured in Fyodorov-Bouchaud [31]. As a matter of fact, Fyodorov-Bouchaud state (1.6.4) as their main result. Mathematically, it is the first occurrence of an explicit formula for the limit density of the properly recentered maximum of a GFF.

A similar story can be told for unitary random matrices. Let U_N denote the $N\times N$ random matrices distributed according to the Haar probability measure on the unitary group $U(N)$. Denoting by $(e^{i\theta_1},\dots,e^{i\theta_n})$ the eigenvalues of U_N, we consider its characteristic polynomial $p_N(\theta)$ evaluated on the unit circle at a point $e^{i\theta}$:

$$p_N(\theta) = \det(1-e^{-i\theta}U_N) = \prod_{k=1}^{N}(1-e^{i(\theta_k-\theta)}). \tag{1.6.5}$$

Recently, the following convergence in law has been obtained in [83] for a real $\alpha \in (-\frac{1}{2}, \sqrt{2})$:

$$\frac{|p_N(\theta)|^\alpha}{\mathbb{E}[|p_N(\theta)|^\alpha]} d\theta \underset{N\to\infty}{\to} e^{\frac{|\alpha|}{2} X(e^{i\theta})} d\theta. \tag{1.6.6}$$

This convergence seems to indicate that $2\ln|p_N(\theta)|$ should be seen as a cut-off of X just like our X_ϵ with N corresponding to $\frac{1}{\epsilon}$. Based on this analogy, it is reasonable that the properly shifted maximum of $2\ln|p_N(\theta)|$ should converge to the same limit as the (properly shifted) maximum of X on the unit circle. Indeed it has been recently conjectured by Fyodorov, Hiary and Keating [32] (and further developed in [33]) that the following convergence in law should hold,

$$\max_{\theta\in[0,2\pi]} \ln|p_N(\theta)| - \ln N + \frac{3}{4}\ln\ln N \underset{N\to\infty}{\to} \mathcal{G}_1 + \mathcal{G}_2 + C, \tag{1.6.7}$$

where $\mathcal{G}_1$ and $\mathcal{G}_2$ are again two independent Gumbel laws and C a real constant. On the mathematical side the most recent result [14] establishes that

$$\max_{\theta\in[0,2\pi]} \ln|p_N(\theta)| - \ln N + \frac{3}{4}\ln\ln N \tag{1.6.8}$$

is tight. Just like for the GFF it is natural to expect that the following convergence is easier to establish directly

$$\max_{\theta\in[0,2\pi]} \ln|p_N(\theta)| - \ln N + \frac{3}{4}\ln\ln N \underset{N\to\infty}{\to} \mathcal{G}_1 + \ln Y' + C. \tag{1.6.9}$$

Our result could then prove instrumental in precisely identifying the limit in the conjectured convergence (1.6.7).

In the case of the unit interval it is also possible with our exact formula to study the behaviour of the maximum of X_I on $[0,1]$ and to establish a link with random Hermitian matrices. The behaviour of the maximum is given by the following convergence in law, first conjectured in [62],

$$\max_{x\in[0,1]} X_{I,\epsilon}(x) - 2\ln\frac{1}{\epsilon} + \frac{3}{2}\ln\ln\frac{1}{\epsilon} \underset{\epsilon\to 0}{\to} \mathcal{G}_1 + \mathcal{G}_2 + \mathcal{N}(0, 4\ln 2) + \ln X_2 + \ln X_3 + C, \tag{1.6.10}$$

with again as before $\mathcal{G}_1$, $\mathcal{G}_2$ two Gumbel laws and C a non-universal constant that depends on the cut-off procedure. On the other hand we see there are three additional terms, a Gaussian law and two additional laws $\ln X_2$ and $\ln X_3$ (see chapter 3 for a precise definition). Moving on to the link with random matrices, to obtain as limit a GMC on the interval the correct matrix ensemble is an ensemble of Hermitian matrices H_N. Their eigenvalues are on the real line but with the right rescaling the limit of the characteristic polynomial will be a GMC on $[0,1]$. We can thus conjecture:

$$\max_{x\in[0,1]} \ln|\det(H_N - x)| - \ln N + \frac{3}{4}\ln\ln N \underset{N\to\infty}{\to} \mathcal{G}_1 + \mathcal{G}_2 + \mathcal{N}(0, 4\ln 2) + \ln X_2 + \ln X_3 + C. \tag{1.6.11}$$

1.6.2 Additional applications

A major motivation for probabilists to study Liouville theory is that it provides the conjectured scaling limit of random planar maps potentially weighted by a statistical physics model. To every integrability formula in Liouville theory there should correspond an observable on random planar maps; see [17] for the conjectures in the case of the Riemann sphere. Let us mention a slightly different approach to Liouville theory developed in [26, 56]. The strength of this approach is that it bridges the gap between the discrete world of random planar maps and the continuum description of Liouville quantum gravity. Major progress has been made in the case of pure gravity $\gamma = \sqrt{\frac{8}{3}}$ corresponding to uniform random planar maps. The same framework has also allowed to derive rigorously the so-called KPZ relation of [44]; see [29]. Establishing links between the approach of [26] and the path integral construction we propose could prove very useful to obtain a better understanding of Liouville theory. Finally, we mention the widespread approach to the probabilistic study of CFT using the celebrated SLE curves. BPZ type equations can be written for these curves [23] and precise links with the Virasoro algebra of CFT have been uncovered [25]. Coupling between the SLE and Liouville theory has also been widely developed in [26].

We can briefly mention a few additional problems linked to Liouville theory. First, the integrability formulas we present in section 1.5 are only a very small part of what can be found in the review [57]. For instance there is another whole set of formulas for the ZZ-branes of Liouville theory. Also, the problem of random geometry has been studied with a modified action where an additional Mabuchi term is added to (1.3.2), see [12]. This new Mabuchi term can be seen as a perturbation of the Liouville CFT. We mention as well the celebrated AGT conjecture [1] linking the Liouville theory to the Nekrasov partition function of a four-dimensional gauge theory. This has been recently studied on the mathematical side in the work [53], where the reflection operator of Liouville theory appears. Finally there is a link between the zeros of the Riemann zeta function and GMC theory, see for instance [75] and references therein.

1.6.3 Integrability program for boundary Liouville theory

In this section we give some perspectives on a series of formulas and results that we expect to obtain with similar techniques. The first conjecture we give is a formula analogue to the Fyodorov-Bouchaud formula but for the bulk measure on the disk $\mathbb{D}$. Its derivation should be quite straightforward by using LCFT on $\mathbb{D}$ but by setting this time $\mu > 0$ and $\mu_\partial = 0$ (instead of the other way around for the Fyodorov-Bouchaud formula).

Conjecture 1. *Let X be the GFF of $\mathbb{D}$ with covariance given by (1.2.6). For $\gamma \in (0,2)$, $\alpha \in (\frac{\gamma}{2}, Q)$, and $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$ we have:*

$$\mathbb{E}[(\int_{\mathbb{D}} \frac{1}{|x|^{\gamma\alpha}} e^{\gamma X} dx^2)^{\frac{Q-\alpha}{\gamma}}] = \gamma^2 \cos(\frac{\alpha - Q}{\gamma}\pi) \frac{\Gamma(\frac{2\alpha}{\gamma} - \frac{4}{\gamma^2})\Gamma(\frac{\gamma}{2}(\alpha - Q))}{\Gamma(\frac{\alpha - Q}{\gamma})} \left(\frac{\pi\Gamma(\frac{\gamma^2}{4})}{\Gamma(1 - \frac{\gamma^2}{4})} \right)^{\frac{Q-\alpha}{\gamma}}. \tag{1.6.12}$$

An intriguing application of the above conjecture is that if we chose $\frac{Q-\alpha}{\gamma} = n$ with $n \in \mathbb{N}$ then we obtain the value of a Selberg type integral on the unit disk that is to the best of our knowledge not known.

Conjecture 2. *Let $\gamma \in (0,2)$ and $n \in \mathbb{N}$ such that $n < \frac{2}{\gamma^2}$. Then the following holds:*

$$\int_{\mathbb{D}} \cdots \int_{\mathbb{D}} dx_1 \dots dx_n \prod_{i=1}^{n} \frac{1}{|x_i|^{2+\frac{\gamma^2}{2}(1-2n)}(1-|x_i|)^{\frac{\gamma^2}{2}}} \prod_{i<j} \frac{1}{|x_i - x_j|^{\gamma^2}|1-\overline{x_i}x_j|^{\gamma^2}} = \frac{\gamma^2 n}{4^n \sqrt{\pi}} \Gamma(-\frac{\gamma^2 n}{2})\Gamma(-n+\frac{1}{2}) \left(\frac{-\pi\Gamma(\frac{\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})}\right)^n. \tag{1.6.13}$$

It can be easily verified that the above conjecture is true for $n = 1$. The second conjecture we discuss is a generalization of the Fyodorov-Bouchaud formula to the case where an arbitrary log-singularity is added. This formula is very similar to the one we proved on the unit interval:

Conjecture 3. *(See [63]) For $\gamma \in (0,2)$, $\beta < Q$ and $p < \frac{4}{\gamma^2} \wedge \frac{2}{\gamma}(Q-\alpha)$,*

$$\mathbb{E}[(\int_0^{2\pi} \frac{1}{|e^{i\theta}-1|^{\frac{\beta\gamma}{2}}} e^{\frac{\gamma}{2}X(e^{i\theta})} d\theta)^p] = (2\pi)^p \frac{\Gamma(1-\frac{\gamma^2 p}{4})\Gamma_\gamma(Q-\beta-\frac{\gamma p}{2})\Gamma_\gamma(Q-\frac{\beta}{2})^2\Gamma_\gamma(Q-\frac{\gamma p}{2})}{\Gamma(1-\frac{\gamma^2}{4})^p\Gamma_\gamma(Q-\frac{\beta}{2}-\frac{\gamma p}{2})^2\Gamma_\gamma(Q-\beta)\Gamma_\gamma(Q)}. \tag{1.6.14}$$

Here the special function Γ_γ is again defined by (1.5.4). In fact the Selberg integral on the circle - the so-called Morris integral - predicts that one can go even furthermore and hope to prove the following formula.

Conjecture 4. *For $\gamma \in (0,2)$ and suitable a, b, $p \in \mathbb{R}$ give a meaning to and prove the following formula:*

$$\mathbb{E}[(\int_0^{2\pi} \frac{e^{\frac{i\theta\gamma}{4}(b-a)}}{|e^{i\theta}-1|^{\frac{\gamma}{2}(a+b)}} e^{\frac{\gamma}{2}X(e^{i\theta})} d\theta)^p] = (2\pi)^p \frac{\Gamma(1-\frac{\gamma^2 p}{4})\Gamma_\gamma(Q-a-b-\frac{\gamma p}{2})\Gamma_\gamma(Q-a)\Gamma_\gamma(Q-b)\Gamma_\gamma(Q-\frac{\gamma p}{2})}{\Gamma(1-\frac{\gamma^2}{4})^p\Gamma_\gamma(Q-a-\frac{\gamma p}{2})\Gamma_\gamma(Q-b-\frac{\gamma p}{2})\Gamma_\gamma(Q-a-b)\Gamma_\gamma(Q)}. \tag{1.6.15}$$

Notice that since we include a complex number inside the expectation it is not straightforward to give a meaning to the above expression. These generalizations of the Fyodorov-Bouchaud formula are the simplest one can think of. Both conjectures 3 and 4 correspond to choosing one of the two cosmological constants μ or μ_∂ equal to 0. But in fact boundary LCFT is of course well-defined when $\mu > 0$ and $\mu_\partial > 0$ in (1.3.19), even if the expression of the correlations (1.3.20) in terms of the GMC measures $e^{\gamma X} dx^2$ and $e^{\frac{\gamma}{2}X} d\theta$ is less straightforward. We will now state a list of formulas is this general case, for convenience we will work on $\mathbb{H}$ just like in the review [57].

Conjecture 5. *For $\mu > 0$, $\mu_\partial > 0$, $z \in \mathbb{H}$, $\alpha \in \mathbb{R}$, prove the following formula for the one-point correlation function of boundary Liouville theory*

$$\langle e^{\alpha\phi(z)} \rangle_{\mathbb{H}} = \frac{U(\alpha)}{|z-\overline{z}|^{2\Delta_\alpha}} \tag{1.6.16}$$

where

$$U(\alpha) = \frac{4}{\gamma}(\pi\mu l(\frac{\gamma^2}{4}))^{\frac{Q-2\alpha}{\gamma}} \Gamma(\frac{\alpha\gamma}{2}-\frac{\gamma^2}{4})\Gamma(\frac{2\alpha}{\gamma}-\frac{4}{\gamma^2}-1)\cosh((\alpha-Q)\pi\nu) \tag{1.6.17}$$

with ν defined by the relation $\cosh^2\frac{\pi\gamma\nu}{2} = \frac{\mu_\partial^2}{\mu}\sin\frac{\pi\gamma^2}{4}$.

This one-point correlation of the boundary Liouville theory is just the first of a set of basic correlation functions that characterize boundary LCFT. More generally one expects to be able to show:

Conjecture 6. *The following correlations have an exact formula given for $z \in \mathbb{H}$, $s, s_1, s_2, s_3 \in \partial\mathbb{H}$, $\alpha, \beta, \beta_1, \beta_2, \beta_3 \in \mathbb{R}$ by,*

$$\langle e^{\frac{\beta_1}{2}\phi(s_1)} e^{\frac{\beta_2}{2}\phi(s_2)} \rangle_{\mathbb{H}} = \frac{d(\beta)}{|s_1 - s_2|^{2\Delta_\beta}}, \tag{1.6.18}$$

$$\langle e^{\alpha\phi(z)} e^{\frac{\beta}{2}\phi(s)} \rangle_{\mathbb{H}} = \frac{R(\alpha,\beta)}{|z-\overline{z}|^{2\Delta_\alpha - \Delta_\beta}|z-s|^{2\Delta_\beta}}, \tag{1.6.19}$$

$$\langle e^{\frac{\beta_1}{2}\phi(s_1)} e^{\frac{\beta_2}{2}\phi(s_2)} e^{\frac{\beta_3}{2}\phi(s_3)} \rangle_{\mathbb{H}} = \frac{c(\beta_1,\beta_2,\beta_3)}{|s_1-s_2|^{\Delta_{12}}|s_2-s_3|^{\Delta_{23}}|s_3-s_1|^{\Delta_{31}}}. \tag{1.6.20}$$

In the review [57] there are conjectured values for the structure constants $d(\beta)$, $R(\alpha,\beta)$, and $c(\beta_1,\beta_2,\beta_3)$ that appear in the above expressions.

Finally let us note that formulas also exist in the case of other topologies (torus [38] or annulus [52]). Proving these formulas is a whole new adventure as one needs to perform the so-called "modular bootstrap". This method is linked to the next subsection.

1.6.4 The conformal bootstrap method of CFT

In the previous sections we have explained how we can obtain the basic correlation functions of Liouville theory on the sphere (1.5.1) or on a domain with boundary (1.6.16), (1.6.18)-(1.6.20). It turns out that using the so-called "conformal bootstrap" method of conformal field theory one can in principle compute recursively all the correlation functions from these basic correlations. A long term research project is to prove the validity of the "conformal bootstrap" method in the well-defined probabilistic framework of LCFT. The first step is to define Liouville correlations with complex parameters, i.e. to give a meaning to the Liouville correlations (1.3.5) and (1.3.20) with complex parameters α_i and β_j. This is a non-trivial problem as one expects to see different phases appear just like in [39]. Once this is well understood one could hope to tackle the following conjecture:

Conjecture 7. *One has the following decomposition of the four-point correlation function of Liouville theory on $\mathbb{S}^2$ as an integral involving the three-point structure constants of (1.5.1), see [71]:*

$$\langle e^{\alpha_0\phi(z)} e^{\alpha_1\phi(0)} e^{\alpha_2\phi(1)} e^{\alpha_3\phi(\infty)} \rangle_{\mathbb{S}^2} = \int_{Q+i\mathbb{R}_+} C_\gamma(\alpha_0,\alpha_1,\alpha) C_\gamma(2Q-\alpha,\alpha_2,\alpha_3) |\mathcal{F}_{\{\alpha_i\}}(z)|^2 d\alpha. \tag{1.6.21}$$

The functions $\mathcal{F}_{\{\alpha_i\}}(z)$ that appear are the so-called universal conformal blocks of CFT.

The mathematical proof of such a decomposition would bridge the gap between the two approaches to Liouville theory: the path integral formulation of (1.3.5) and the algebraic bootstrap method of [71]. In the trivial case where we choose $\mu = 0$ in (1.3.3) the decomposition (1.6.21) is simply a Fourier transform; therefore the case $\mu > 0$ can be seen as a sort of generalized Fourier decomposition (see [45]). In the case of a domain with boundary, a similar decomposition is expected to hold [81].

Conjecture 8. *A decomposition similar to (1.6.21) also holds for the boundary Liouville theory. This conjecture is discussed in more details in [81].*

Both conjectures 7 and 8 could prove to be quite difficult. An easier starting point could be to chose again $\mu = 0$ in boundary LCFT in order to write everything only for a GMC measure on the unit circle. Thus one could attempt to show a similar decomposition for a very general form of the Fyodorov-Bouchaud formula (1.5.9) with an arbitrary number of log-singularities.

Conjecture 9. *For $z_1, \dots, z_N \in \mathbb{D}$, $s_1, \dots, s_N \in \partial\mathbb{D}$, $\alpha_i, \beta_j \in \mathbb{C}$, decompose the following function*

$$G(\gamma, \alpha_i, \beta_j, p) = \mathbb{E}[(\int_0^{2\pi} \prod_{i=1}^{N} \frac{1}{|e^{i\theta} - z_i|^{\alpha_i \gamma}} \prod_{j=1}^{M} \frac{1}{|e^{i\theta} - s_j|^{\frac{\beta_j \gamma}{2}}} e^{\frac{\gamma}{2} X(e^{i\theta})} d\theta)^p]$$

as a sum of other $G(\gamma, \alpha_i, \beta_j, p)$ but with fewer insertion points z_i, s_j.

Chapter 2

The Fyodorov-Bouchaud formula

In a remarkable paper in 2008, Fyodorov and Bouchaud conjectured an exact formula for the density of the total mass of (sub-critical) Gaussian multiplicative chaos (GMC) associated to the Gaussian free field (GFF) on the unit circle [31]. In this paper we will give a proof of this formula. In the mathematical literature this is the first occurrence of an explicit probability density for the total mass of a GMC measure. The key observation of our proof is that the negative moments of the total mass of GMC determine its law and are equal to one-point correlation functions of Liouville conformal field theory in the disk defined by Huang, Rhodes and Vargas [40]. The rest of the proof then consists in implementing rigorously the framework of conformal field theory (BPZ equations for degenerate field insertions) in a probabilistic setting to compute the negative moments. Finally we will discuss applications to random matrix theory, asymptotics of the maximum of the GFF and tail expansions of GMC.

2.1 Introduction and main result

Starting from a Gaussian free field (GFF) one can by standard regularization techniques define the associated Gaussian multiplicative chaos (GMC) measure whose density is formally given by the exponential of the GFF. The theory of GMC goes back to Kahane's 1985 paper [42] and has grown into an important field within probability theory and mathematical physics with applications to 3d turbulence, mathematical finance, extreme values of log-correlated processes, disordered systems, random geometry and 2d quantum gravity. See for instance [72] for a review.

In this chapter we will be concerned with the last application and more precisely with the link between GMC and the correlation functions of Liouville conformal field theory (LCFT). It is this connection uncovered in 2014 in [17] that enables us to understand the integrability of GMC measures and perform exact computations. The very recent proof of the DOZZ formula [46, 47] can be seen as the first integrability result on fractional moments of GMC measures while our Theorem 2.1.1 is the first result that gives an explicit probability density for the total mass of a GMC measure.

We will now introduce the framework of our paper. Let X be a GFF on the unit disk $\mathbb{D}$ with covariance given for $x, y \in \mathbb{D}$ by:[1]

$$\mathbb{E}[X(x)X(y)] = \ln \frac{1}{|x-y||1-x\overline{y}|}. \tag{2.1.1}$$

[1]This is the GFF with Neumann boundary conditions also called the free boundary conditions, see [40].

In the case of two points $e^{i\theta}$ and $e^{i\theta'}$ on the unit circle $\partial\mathbb{D}$, this simply reduces to:[2]

$$\mathbb{E}[X(e^{i\theta})X(e^{i\theta'})] = 2\ln\frac{1}{|e^{i\theta}-e^{i\theta'}|}. \tag{2.1.2}$$

In this setting and for all $\gamma \geq 0$, the GMC measure on the unit circle is constructed as the following limit in probability in the sense of weak convergence of measures,

$$e^{\frac{\gamma}{2}X(e^{i\theta})}d\theta := \lim_{\epsilon\to 0} e^{\frac{\gamma}{2}X_\epsilon(e^{i\theta})-\frac{\gamma^2}{8}\mathbb{E}[X_\epsilon(e^{i\theta})^2]}d\theta, \tag{2.1.3}$$

where $d\theta$ is the Lebesgue measure on $[0,2\pi]$ and X_ϵ is a reasonable cut-off approximation of X which converges to X as ϵ goes to 0. More precisely for any continuous test function $f : \partial\mathbb{D} \mapsto \mathbb{R}$, the following holds in probability:

$$\int_0^{2\pi} e^{\frac{\gamma}{2}X(e^{i\theta})}f(e^{i\theta})d\theta = \lim_{\epsilon\to 0}\int_0^{2\pi} e^{\frac{\gamma}{2}X_\epsilon(e^{i\theta})-\frac{\gamma^2}{8}\mathbb{E}[X_\epsilon(e^{i\theta})^2]}f(e^{i\theta})d\theta. \tag{2.1.4}$$

See for instance Berestycki [9] for an elegant proof of this convergence. It goes back to Kahane [42] that the measure $e^{\frac{\gamma}{2}X(e^{i\theta})}d\theta$ defined by (2.1.3) is different from 0 if and only if $\gamma \in [0,2)$. In the sequel, we will always work with $\gamma \in (0,2)$ (with the exception of section 2.1.1.1 where we will discuss the limit $\gamma \to 2$). We now introduce the main quantity of interest of our paper, the partition function of the theory, for $\gamma \in (0,2)$:

$$Y_\gamma = \frac{1}{2\pi}\int_0^{2\pi} e^{\frac{\gamma}{2}X(e^{i\theta})}d\theta. \tag{2.1.5}$$

Recall the following classical fact on the existence of moments for GMC (see the reviews by Rhodes-Vargas [72, 73] for instance), for $p \in \mathbb{R}$:

$$\mathbb{E}[Y_\gamma^p] < +\infty \iff p < \frac{4}{\gamma^2}. \tag{2.1.6}$$

In 2008 Fyodorov and Bouchaud [31] conjectured an exact formula for the density of Y_γ. Their conjecture is based on the computation of the integer moments of Y_γ and a clever observation. If X_ϵ is a reasonable cut-off approximation of X then for all p *nonnegative integer* such that $p < \frac{4}{\gamma^2}$ one gets by Fubini (interchanging $\int_0^{2\pi}$ and $\mathbb{E}[.]$):

$$\begin{aligned}\mathbb{E}[(\frac{1}{2\pi}\int_0^{2\pi} e^{\frac{\gamma}{2}X_\epsilon(e^{i\theta})-\frac{\gamma^2}{8}\mathbb{E}[X_\epsilon(e^{i\theta})^2]}d\theta)^p] &= \frac{1}{(2\pi)^p}\int_{[0,2\pi]^p}\mathbb{E}[\prod_{i=1}^p e^{\frac{\gamma}{2}X_\epsilon(e^{i\theta_i})-\frac{\gamma^2}{8}\mathbb{E}[X_\epsilon(e^{i\theta_i})^2]}]d\theta_1\dots d\theta_p \\ &= \frac{1}{(2\pi)^p}\int_{[0,2\pi]^p} e^{\frac{\gamma^2}{4}\sum_{i<j}\mathbb{E}[X_\epsilon(e^{i\theta_i})X_\epsilon(e^{i\theta_j})]}d\theta_1\dots d\theta_p\end{aligned}$$

By taking the limit in the above computation as ϵ goes to 0 one gets:

$$\begin{aligned}\mathbb{E}[Y_\gamma^p] &= \frac{1}{(2\pi)^p}\int_{[0,2\pi]^p} e^{\frac{\gamma^2}{4}\sum_{i<j}\mathbb{E}[X(e^{i\theta_i})X(e^{i\theta_j})]}d\theta_1\dots d\theta_p \\ &= \frac{1}{(2\pi)^p}\int_{[0,2\pi]^p}\prod_{i<j}\frac{1}{|e^{i\theta_i}-e^{i\theta_j}|^{\frac{\gamma^2}{2}}}d\theta_1\dots d\theta_p.\end{aligned}$$

[2]This normalization is different from the $\ln\frac{1}{|e^{i\theta}-e^{i\theta'}|}$ usually found in the literature.

The main observation of Fyodorov and Bouchaud [31] is that the last integral above is a circular variant of the famous Selberg integral, the so-called Morris integral, and its value is explicitly known as $\frac{\Gamma(1-p\frac{\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})^p}$. Hence, one gets for p nonnegative integer such that $p < \frac{4}{\gamma^2}$:

$$\mathbb{E}[Y_\gamma^p] = \frac{\Gamma(1-p\frac{\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})^p}. \tag{2.1.7}$$

Fyodorov and Bouchaud then conjectured that the identity (2.1.7) remains valid if p is any real number such that $p < \frac{4}{\gamma^2}$. Though the conjecture is reasonable, one should notice that it is far from obvious. Indeed both sides of (2.1.7) are analytic functions of p equal on the finite set $\{0, 1, \cdots, \lfloor \frac{4}{\gamma^2} \rfloor\}$ (where $\lfloor . \rfloor$ denotes integer part) and this does not guarantee that they are equal on their domain of definition. The main result of this paper is precisely to prove this point:

Theorem 2.1.1. *(Fyodorov-Bouchaud formula) Let $\gamma \in (0,2)$. For all real[3] p such that $p < \frac{4}{\gamma^2}$ the following identity holds:*

$$\mathbb{E}[Y_\gamma^p] = \frac{\Gamma(1-p\frac{\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})^p}. \tag{2.1.8}$$

As a consequence, the variable Y_γ has an explicit density given by[4]

$$f_{Y_\gamma}(y) = \frac{4\beta}{\gamma^2}(\beta y)^{-\frac{4}{\gamma^2}-1} e^{-(\beta y)^{-\frac{4}{\gamma^2}}} \mathbf{1}_{[0,\infty[}(y) \tag{2.1.9}$$

where we have set $\beta = \Gamma(1-\frac{\gamma^2}{4})$.

Let us make a few comments on the above density. The probability for Y_γ to be large is governed by the term $(\beta y)^{-\frac{4}{\gamma^2}-1}$. The power $-\frac{4}{\gamma^2}-1$ is of course compatible with the existence of moments (2.1.6) and is a very universal feature of GMC measures. For instance it holds for more general covariances (see section 2.1.1.3 for more on the tail behaviour of GMC) and it holds for GMC on the unit interval $[0,1]$. On the other hand the probability for Y_γ to be small is given by the term $\exp(-(\beta y)^{-\frac{4}{\gamma^2}})$ which is extremely small and implies that the negative moments of Y_γ determine its law, a key ingredient of our proof. This behaviour is quite mysterious and model specific as it differs from the case of a GMC on $[0,1]$ and differs from the case of a log-normal law, where instead we would have $\exp(-\ln(y)^2)$.

Before explaining the main ideas behind the proof and the connection with Liouville conformal field theory, we will first enumerate the numerous applications of this result.

2.1.1 Applications

2.1.1.1 Critical GMC and the maximum of the GFF on the circle

A problem that has attracted a lot of attention is the behaviour of the maximum of the Gaussian free field on the unit circle. See for instance [4] for a review on extreme value statistics of log-

[3]The result also holds for all complex p such that $\mathrm{Re}(p) < \frac{4}{\gamma^2}$.

[4]Following a remark by Nicolas Curien, a third way of stating our theorem would be to say that $Y_\gamma \overset{law}{=} \frac{1}{\beta} Z^{-\frac{\gamma^2}{4}}$ where Z is an exponential law of parameter 1. It is not clear whether this exponential law has a probabilistic interpretation.

correlated processes. The link with GMC theory goes as follows, it is possible to make sense of GMC in the critical case $\gamma = 2$ by the so-called derivative martingale construction. In this case, the measure denoted by $-\frac{1}{2}X(e^{i\theta})e^{X(e^{i\theta})}d\theta$ is obtained as the following limit,

$$-\frac{1}{2}X(e^{i\theta})e^{X(e^{i\theta})}d\theta := -\lim_{\epsilon\to 0}(\frac{1}{2}X_\epsilon(e^{i\theta}) - \frac{1}{2}\mathbb{E}[X_\epsilon(e^{i\theta})^2])e^{X_\epsilon(e^{i\theta})-\frac{1}{2}\mathbb{E}[X_\epsilon(e^{i\theta})^2]}d\theta, \tag{2.1.10}$$

where X_ϵ is a reasonable cut-off approximation of X which converges to X as ϵ goes to 0. The construction (2.1.10) converges to a non trivial random positive measure. This was proved in [27, 28] for specific cut-offs X_ϵ and generalized to general cut-offs in [68]. We now introduce:

$$Y' := -\frac{1}{2}\int_0^{2\pi} X(e^{i\theta})e^{X(e^{i\theta})}d\theta. \tag{2.1.11}$$

It is natural to expect that Y' can be obtained from the sub-critical measures Y_γ as γ goes to 2 by taking a suitable limit. Indeed it is shown in [6] that the following holds in probability:[5]

$$2Y' = \lim_{\gamma\to 2}\frac{1}{2-\gamma}Y_\gamma. \tag{2.1.12}$$

From this convergence and Theorem 2.1.1, one can deduce that $2Y'$ has a density $f_{2Y'}$ given by

$$f_{2Y'}(y) = y^{-2}e^{-y^{-1}}\mathbf{1}_{[0,\infty[}(y). \tag{2.1.13}$$

We observe that $\ln 2Y'$ is distributed like a standard Gumbel law. Recall that an impressive series of works (see [13, 15] for the latest results) have proven that for suitable sequences of cut-off approximations X_ϵ the following convergence in law holds[6]

$$\max_{\theta\in[0,2\pi]} X_\epsilon(e^{i\theta}) - 2\ln\frac{1}{\epsilon} + \frac{3}{2}\ln\ln\frac{1}{\epsilon} \underset{\epsilon\to 0}{\to} \mathcal{G} + \ln Y' + C \tag{2.1.14}$$

where $\mathcal{G}$ is a Gumbel law independent from Y' and C is a non universal constant that depends on the cut-off procedure. From this convergence and previous considerations, one can deduce the following convergence in law

$$\max_{\theta\in[0,2\pi]} X_\epsilon(e^{i\theta}) - 2\ln\frac{1}{\epsilon} + \frac{3}{2}\ln\ln\frac{1}{\epsilon} \underset{\epsilon\to 0}{\to} \mathcal{G}_1 + \mathcal{G}_2 + C \tag{2.1.15}$$

where $\mathcal{G}_1$ and $\mathcal{G}_2$ are two independent Gumbel laws and where we have absorbed the factor $\ln 2$ in the constant C. This convergence was conjectured in Fyodorov-Bouchaud [31]. As a matter of fact, Fyodorov-Bouchaud state (2.1.15) as their main result.[7] Mathematically, it is the first occurrence of an explicit formula for the limit density of the properly recentered maximum of a GFF.

[5] In [6] the convergence is actually written for the two-dimensional measure but the authors are confident that their method still works in dimension 1.

[6] In fact, the works [13, 15] establish the convergence result (2.1.14) with a variable Y' which has not yet been rigorously proved to be the same as our definition (2.1.11) of Y'. Nonetheless, private communications with the authors of [13, 15] confirm that their methods can be extended to prove that both definitions of Y' coincide.

[7] More accurately they expressed the limit density in terms of a modified Bessel function which was noticed by Subag and Zeitouni in [79] to be the sum of two independent Gumbel laws.

2.1.1.2 Unitary random matrix theory

A similar story can be told for unitary random matrices. Let U_N denote the $N \times N$ random matrices distributed according to the Haar probability measure on the unitary group $U(N)$. Denoting by $(e^{i\theta_1}, \dots, e^{i\theta_n})$ the eigenvalues of U_N, we consider its characteristic polynomial $p_N(\theta)$ evaluated on the unit circle at a point $e^{i\theta}$:

$$p_N(\theta) = \det(1 - e^{-i\theta} U_N) = \prod_{k=1}^{N} (1 - e^{i(\theta_k - \theta)}). \tag{2.1.16}$$

Recently, the following convergence in law has been obtained in [83] for a real $\alpha \in (-\frac{1}{2}, \sqrt{2})$:

$$\frac{|p_N(\theta)|^\alpha}{\mathbb{E}[|p_N(\theta)|^\alpha]} d\theta \underset{N\to\infty}{\to} e^{\frac{|\alpha|}{2} X(e^{i\theta})} d\theta. \tag{2.1.17}$$

This convergence seems to indicate that $2 \ln |p_N(\theta)|$ should be seen as a cut-off of X just like our X_ϵ with N corresponding to $\frac{1}{\epsilon}$. We thus expect to have for a real $p < \frac{4}{\alpha^2}$:[8]

$$\mathbb{E}[(\frac{1}{2\pi} \int_0^{2\pi} \frac{|p_N(\theta)|^\alpha}{\mathbb{E}[|p_N(\theta)|^\alpha]} d\theta)^p] \underset{N\to\infty}{\to} \mathbb{E}[(\frac{1}{2\pi} \int_0^{2\pi} e^{\frac{|\alpha|}{2} X(e^{i\theta})} d\theta)^p] = \frac{\Gamma(1 - p\frac{\alpha^2}{4})}{\Gamma(1 - \frac{\alpha^2}{4})^p} \tag{2.1.18}$$

Notice that $\mathbb{E}[|p_N(\theta)|^\alpha]$ is independent of θ. Now the following asymptotic is known for $\alpha > -1$,

$$\mathbb{E}[|p_N(\theta)|^\alpha] \underset{N\to\infty}{\sim} \frac{G(1+\alpha/2)^2}{G(1+\alpha)} N^{\frac{\alpha^2}{4}}, \tag{2.1.19}$$

where G is the so-called Barnes' function. Combining (2.1.18) with (2.1.19) establishes the asymptotic conjectured in [32] and further studied in [49], for a real $p < \frac{4}{\alpha^2}$:

$$\mathbb{E}[(\frac{1}{2\pi} \int_0^{2\pi} |p_N(\theta)|^\alpha d\theta)^p] \underset{N\to\infty}{\sim} \frac{G(1+\alpha/2)^{2p}}{G(1+\alpha)^p} \frac{\Gamma(1 - p\frac{\alpha^2}{4})}{\Gamma(1 - \frac{\alpha^2}{4})^p} N^{\frac{p\alpha^2}{4}}. \tag{2.1.20}$$

Now again based on the analogy suggested by (2.1.17), it is reasonable that the properly shifted maximum of $2 \ln |p_N(\theta)|$ should converge to the same limit as the (properly shifted) maximum of the GFF on the circle. Indeed it has been recently conjectured by Fyodorov, Hiary and Keating [32] that the following convergence in law should hold

$$\max_{\theta \in [0, 2\pi]} \ln |p_N(\theta)| - \ln N + \frac{3}{4} \ln \ln N \underset{N\to\infty}{\to} \mathcal{G}_1 + \mathcal{G}_2 + C \tag{2.1.21}$$

where $\mathcal{G}_1$ and $\mathcal{G}_2$ are again two independent Gumbel laws and C a real constant. On the mathematical side, there has been a series of works [5, 64, 14] aiming at this result. The most recent result [14] establishes that

$$\max_{\theta \in [0, 2\pi]} \ln |p_N(\theta)| - \ln N + \frac{3}{4} \ln \ln N \tag{2.1.22}$$

[8] This convergence of moments has never rigorously been written down but in the L^2 phase, i.e. for $|\alpha| < \sqrt{2}$, it is more or less a consequence of [83]. Going beyond the L^2 phase seems to require much more technical work similar to the techniques of [9] to define GMC outside the L^2 phase.

is tight. Just like for the GFF it is natural to expect that the following convergence is easier to establish directly

$$\max_{\theta\in[0,2\pi]} \ln|p_N(\theta)| - \ln N + \frac{3}{4}\ln\ln N \underset{N\to\infty}{\to} \mathcal{G}_1 + \ln Y' + C. \quad (2.1.23)$$

Our result could then prove instrumental in precisely identifying the limit in the conjectured convergence (2.1.21).

2.1.1.3 The tail of GMC in dimension 1

Finally, Rhodes and Vargas in [74] introduced a simple method to compute tail expansions for general GMC measures. The authors claim the method works for GMC measures in dimension 1 and 2 associated to any log-correlated field (the method probably works in all dimensions). More precisely, in the 1d case, consider a log-correlated field $\tilde{X}$ on an open set $\mathcal{O} \subset \partial\mathbb{D}$ with the following covariance,

$$\mathbb{E}[\tilde{X}(e^{i\theta})\tilde{X}(e^{i\theta'})] = 2\ln\frac{1}{|e^{i\theta}-e^{i\theta'}|} + f(e^{i\theta}, e^{i\theta'}), \quad (2.1.24)$$

for a smooth function f. The authors of [74] argued that the following should hold for some $\delta > 0$,

$$\mathbb{P}\left(\int_{\mathcal{O}} e^{\frac{\gamma}{2}\tilde{X}(e^{i\theta})}d\theta > t\right) \underset{t\to\infty}{=} \left(\int_{\mathcal{O}} e^{(\frac{4}{\gamma^2}-1)f(e^{i\theta},e^{i\theta})}d\theta\right)\left(1-\frac{\gamma^2}{4}\right)\frac{R_1(\gamma)}{t^{\frac{4}{\gamma^2}}} + o(t^{-\frac{4}{\gamma^2}-\delta}), \quad (2.1.25)$$

where $\mathcal{O}$ is an open subset of $\partial\mathbb{D}$ and $R_1(\gamma)$ is a non explicit universal constant defined in terms of the expectation of a random variable. Since Theorem 2.1.1 gives an explicit tail expansion for Y_γ and the variable Y_γ has a tail expansion which satisfies (2.1.25), one can deduce an explicit value for $R_1(\gamma)$. This leads to

$$R_1(\gamma) = \frac{(2\pi)^{\frac{4}{\gamma^2}-1}}{(1-\frac{\gamma^2}{4})\Gamma(1-\frac{\gamma^2}{4})^{\frac{4}{\gamma^2}}}. \quad (2.1.26)$$

2.1.2 Strategy of the proof

To explain the ideas behind our proof of Theorem 2.1.1, we must make a detour in the world of conformal field theory (CFT). In 2014, David, Kupiainen, Rhodes and Vargas [17] applied the theory of GMC to define rigorously Liouville conformal field theory (LCFT) on the Riemann sphere. This theory was first introduced by A. Polyakov in his 1981 seminal paper [66] where he proposed a path integral theory of random two dimensional surfaces. In [17] the authors discovered that the correlation functions of LCFT could be expressed as fractional moments of GMC measures with log singularities therefore rendering possible the mathematical study of LCFT. The theory was defined on the Riemann sphere in [17] then on the unit disk in [40] and on other surfaces in [18], [69]. Let us also mention another interesting approach by Duplantier-Miller-Sheffield [26] which develops a theory of quantum surfaces with two marked points linked to the two-point correlation function of LCFT (see [82] for a precise statement of this connection).

Since Liouville theory is a CFT, one expects that it is possible to use the framework developed by Belavin, Polyakov and Zamolodchikov (BPZ) in [8] to compute explicitly its correlation functions. As a matter of fact, the original motivation of BPZ for introducing CFT was to compute the correlations of LCFT although it has now grown into a huge field of theoretical physics. Recently, Kupiainen, Rhodes and Vargas were indeed able to rigorously implement the BPZ framework for LCFT in a probabilistic setting. As an output of their constructions, they gave a proof of the celebrated DOZZ formula [46, 47] for the three-point function of LCFT whose value was conjectured independently by Dorn and Otto in [22] and by Zamolodchikov and Zamolodchikov in [86].

Concerning the strategy of our proof, the key observation is to realize that the inverse moments of GMC integrated on the unit circle can be expressed as one-point correlation functions of LCFT on the unit disk. This link was to the best of our knowledge unknown even to physicists. Thanks to this observation, we can develop the framework of CFT to compute the inverse moment using a strategy similar to the proof of the DOZZ formula [46, 47]. However, working on a domain with boundary requires to introduce a novel BPZ differential equation - see Theorem 2.2.2 below - which differs from the equation of [46] on the Riemann sphere.

Let us now introduce some notations which will be used in the sequel. Let p be a real number such that $p < \frac{4}{\gamma^2}$. We denote:

$$U(\gamma, p) = \mathbb{E}[(\int_0^{2\pi} e^{\frac{\gamma}{2}X(e^{i\theta})} d\theta)^p]. \tag{2.1.27}$$

The proof is based on studying the following function or "observable" defined for $t \in [0, 1]$:

$$G(\gamma, p, t) = \mathbb{E}[(\int_0^{2\pi} |t - e^{i\theta}|^{\frac{\gamma^2}{2}} e^{\frac{\gamma}{2}X(e^{i\theta})} d\theta)^p]. \tag{2.1.28}$$

At first glance, it can seem mysterious why the introduction of $G(\gamma, p, t)$ can be of any help in computing $U(\gamma, p)$. In order to understand why $G(\gamma, p, t)$ is the "right" auxiliary function to look at, one must cast the problem in the language of LCFT with boundary. It turns out that the moment $U(\gamma, p)$ is the one-point correlation function and that the function $G(\gamma, p, t)$ is the two-point correlation function with a so-called degenerate field insertion, see section 2.2 for the definitions. Therefore the function $G(\gamma, p, t)$ is expected to obey a differential equation known as the BPZ equation. Indeed, we will prove using probabilistic techniques that:

Proposition 2.1.2. *(BPZ) For $\gamma \in (0, 2)$ and $p < 0$ the function $t \mapsto G(\gamma, p, t)$ satisfies the following differential equation:*

$$(t(1-t^2)\frac{\partial^2}{\partial t^2} + (t^2 - 1)\frac{\partial}{\partial t} + 2(C - (A + B + 1)t^2)\frac{\partial}{\partial t} - 4ABt)G(\gamma, p, t) = 0$$

with the following values for A, B, and C:

$$A = -\frac{\gamma^2 p}{4}, \; B = -\frac{\gamma^2}{4}, \; C = \frac{\gamma^2}{4}(1-p) + 1.$$

Here the hypothesis on p is purely technical and could be relaxed with little effort. A simple change of variable $x = t^2$ and $G(\gamma, p, t) = H(x)$ turns the BPZ equation for $G(\gamma, p, t)$ into a hypergeometric equation for $H(x)$

$$(x(1-x)\frac{\partial^2}{\partial x^2} + (C - (A + B + 1)x)\frac{\partial}{\partial x} - AB)H(x) = 0.$$

The solution space of this equation is two dimensional. In fact, we can give two sets of solutions, one corresponding to an expansion in powers of x and the other to an expansion in powers of $1 - x$ (all the details are written in the appendix). From this we obtain:

Proposition 2.1.3. *For $\gamma \in (0,2)$ and $p < 0$, we have*

$$G(\gamma,p,t) = C_1 F(-\frac{\gamma^2 p}{4}, -\frac{\gamma^2}{4}, \frac{\gamma^2}{4}(1-p)+1, t^2) + C_2 t^{\frac{\gamma^2}{2}(p-1)} F(-\frac{\gamma^2}{4}, \frac{\gamma^2}{4}(p-2), \frac{\gamma^2}{4}(p-1)+1, t^2)$$

and

$$G(\gamma,p,t) = B_1 F(-\frac{\gamma^2 p}{4}, -\frac{\gamma^2}{4}, -\frac{\gamma^2}{2}, 1-t^2) + B_2 (1-t^2)^{1+\frac{\gamma^2}{2}} F(1+\frac{\gamma^2}{4}, \frac{\gamma^2}{4}(2-p)+1, 2+\frac{\gamma^2}{2}, 1-t^2)$$

where F is the standard hypergeometric series. The coefficients C_1, C_2, B_1 and B_2 are real constants that depend on γ and p. Since the solution space of the hypergeometric equation is two dimensional, the coefficients are linked by the explicit change of basis formula (2.4.9) written in the appendix.

The end of the proof is based on exploiting the fact that the above coefficients C_1, C_2, B_1 and B_2 can be identified in terms of $U(\gamma,p)$ by performing asymptotic expansions directly on the expression (2.1.28) of $G(\gamma,p,t)$. Notice for instance that $C_1 = G(\gamma,p,0) = U(\gamma,p)$. We also express B_2 in terms of $U(\gamma,p-1)$ and find $C_2 = 0$. Using the change of basis formula (2.4.9) this leads to the following shift equation for $U(\gamma,p)$:

Proposition 2.1.4. *For all $\gamma \in (0,2)$ and for $p \leq 0$, we have the relation:*

$$U(\gamma,p) = \frac{2\pi\Gamma(1-p\frac{\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})\Gamma(1-(p-1)\frac{\gamma^2}{4})} U(\gamma,p-1).$$

From this shift equation we deduce recursively all the positive moments of the variable $\frac{1}{Y_\gamma}$, i.e. we get

$$\mathbb{E}[Y_\gamma^{-n}] = \Gamma(1+\frac{n\gamma^2}{4})\Gamma(1-\frac{\gamma^2}{4})^n, \quad \forall n \in \mathbb{N}. \tag{2.1.29}$$

The series

$$\lambda \mapsto \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\Gamma(1+\frac{n\gamma^2}{4})\Gamma(1-\frac{\gamma^2}{4})^n$$

has an infinite radius of convergence, meaning that the moments of $\frac{1}{Y_\gamma}$ entirely determine its law and one can even give an explicit probability density for $\frac{1}{Y_\gamma}$,

$$f_{\frac{1}{Y_\gamma}}(y) = \frac{4}{\beta\gamma^2}(\frac{y}{\beta})^{\frac{4}{\gamma^2}-1} e^{-(\frac{y}{\beta})^{\frac{4}{\gamma^2}}} \mathbf{1}_{[0,\infty[}(y), \tag{2.1.30}$$

where $\beta = \Gamma(1-\frac{\gamma^2}{4})$. It can easily be turned into a probability density for Y_γ,

$$f_{Y_\gamma}(y) = \frac{4\beta}{\gamma^2}(\beta y)^{-\frac{4}{\gamma^2}-1} e^{-(\beta y)^{-\frac{4}{\gamma^2}}} \mathbf{1}_{[0,\infty[}(y), \tag{2.1.31}$$

which proves Theorem 2.1.1. Also notice that our proof does not use the value of the Morris integral (2.1.7) and in fact we give a new proof of its value by taking integer moments in our GMC measure.

Lastly, we point out that we have actually done more than just compute $U(\gamma,p)$, we have also completely determined the function $G(\gamma,p,t)$. By choosing $t=1$ we obtain the following corollary:

Corollary 2.1.5. *Let $\gamma \in (0,2)$. For all real p such that $p < \frac{4}{\gamma^2}$ the following identity holds:*

$$\mathbb{E}[(\frac{1}{2\pi}\int_0^{2\pi} |1-e^{i\theta}|^{\frac{\gamma^2}{2}} e^{\frac{\gamma}{2}X(e^{i\theta})} d\theta)^p] = \frac{\Gamma(1-p\frac{\gamma^2}{4})\Gamma(1+\frac{\gamma^2}{2})\Gamma(1+(1-p)\frac{\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})^p\Gamma(1+\frac{\gamma^2}{4})\Gamma(1+(2-p)\frac{\gamma^2}{4})}. \tag{2.1.32}$$

Equivalently we also have

$$\frac{1}{2\pi}\int_0^{2\pi} |1-e^{i\theta}|^{\frac{\gamma^2}{2}} e^{\frac{\gamma}{2}X(e^{i\theta})} d\theta \overset{law}{=} Y_\gamma X_1^{-\frac{\gamma^2}{4}} \tag{2.1.33}$$

with Y_γ, X_1 independent and $X_1 \sim \mathcal{B}(1+\frac{\gamma^2}{4}, \frac{\gamma^2}{4})$, where $\mathcal{B}(\alpha,\beta)$ denotes the standard beta law.

A similar formula is also expected to hold for the so-called dual degenerate insertion:

Conjecture 10. *Let $\gamma \in (0,2)$. For all real p such that $p < \frac{4}{\gamma^2}$ we expect to have*

$$\mathbb{E}[(\frac{1}{2\pi}\int_0^{2\pi} |1-e^{i\theta}|^{2} e^{\frac{\gamma}{2}X(e^{i\theta})} d\theta)^p] = \frac{\Gamma(1-p\frac{\gamma^2}{4})\Gamma(1+\frac{8}{\gamma^2})\Gamma(1+\frac{4}{\gamma^2}-p)}{\Gamma(1-\frac{\gamma^2}{4})^p\Gamma(1+\frac{4}{\gamma^2})\Gamma(1+\frac{8}{\gamma^2}-p)} \tag{2.1.34}$$

and we can write again

$$\frac{1}{2\pi}\int_0^{2\pi} |1-e^{i\theta}|^{2} e^{\frac{\gamma}{2}X(e^{i\theta})} d\theta \overset{law}{=} Y_\gamma X_2^{-1} \tag{2.1.35}$$

with Y_γ, X_2 independent and $X_2 \sim \mathcal{B}(1+\frac{4}{\gamma^2}, \frac{4}{\gamma^2})$.

In fact we expect that using similar techniques it will be possible to obtain many more exact formulas on GMC measures. One could study more general cases on the unit circle (such as Conjecture 10 or the conjectures of [62]) or GMC on different geometries such as the unit interval $[0,1]$[9] or the two-dimensional measure on the unit disk $\mathbb{D}$. Therefore our methods combined with the proof of the DOZZ formula [46, 47] open up brand new perspectives for studying the integrability of GMC measures.

2.2 Boundary Liouville Conformal Field Theory

2.2.1 The Liouville correlation functions on $\mathbb{H}$

In order to prove Proposition 2.1.2 we introduce the framework of LCFT on a domain with boundary, following the setting of [40]. Here we will work on the upper half plane $\mathbb{H}$ (with boundary $\partial\mathbb{H} = \mathbb{R}$) but we can transpose everything easily to the unit disk $\mathbb{D}$ by the KPZ relation (2.2.12). The starting point is the well known Liouville action where in our case we must add a boundary term,

$$S_L(X,\hat{g}) = \frac{1}{4\pi}\int_{\mathbb{H}} (|\partial^{\hat{g}} X|^2 + QR_{\hat{g}}X)\hat{g}(z)dz^2 + \frac{1}{2\pi}\int_{\mathbb{R}} (QK_{\hat{g}}X + 2\pi\mu_\partial e^{\frac{\gamma}{2}X})\hat{g}(s)^{1/2}ds, \text{[10]} \tag{2.2.1}$$

[9] Work in progress with Tunan Zhu.

[10] The action usually also contains a bulk interaction term $\mu e^{\gamma X}$ but for our purposes we set $\mu = 0$. Hence we are working with a degenerate form of boundary LCFT.

where $\partial^{\hat{g}}$, $R_{\hat{g}}$, and $K_{\hat{g}}$ respectively stand for the gradient, Ricci scalar curvature and geodesic curvature of the boundary in the metric $\hat{g}$ (which can be chosen arbitrarily). We also have $\gamma \in (0,2)$, $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$, and $\mu_\partial > 0$. In the following we choose the background metric $\hat{g}(z) = \frac{4}{|z+i|^4}$ on $\mathbb{H}$. This is a natural choice as if we map $\mathbb{H}$ to $\mathbb{D}$ (using $z \mapsto \frac{z-i}{z+i}$) this choice corresponds to the Euclidean (or flat) metric on $\mathbb{D}$. With our choice of $\hat{g}$ we get $R_{\hat{g}} = 0$ and $K_{\hat{g}} = 1$ which simplifies the expression of (2.2.1).

With this action we can formally define the correlation functions of LCFT. They are the quantities of interest of the theory that we hope to be able to compute with the techniques of CFT. We will consider two types of insertion points in our correlations: bulk insertions (z_i, α_i) (with $z_i \in \mathbb{H}$ and $\alpha_i \in \mathbb{R}$) and boundary insertions (s_j, β_j) (with $s_j \in \mathbb{R}$ and $\beta_j \in \mathbb{R}$). We introduce the following notations for the so-called vertex operators:

$$V_{\alpha_i}(z_i) = e^{\alpha_i (X(z_i) + \frac{Q}{2} \ln \hat{g}(z_i))}$$
$$V_{\beta_j}(s_j) = e^{\frac{\beta_j}{2} (X(s_j) + \frac{Q}{2} \ln \hat{g}(s_j))}.$$

We formally define the correlations by,

$$\langle \prod_{i=1}^N V_{\alpha_i}(z_i) \prod_{j=1}^M V_{\beta_j}(s_j) \rangle_{\mathbb{H}} = \int_\Sigma D_{\hat{g}} X \prod_{i=1}^N e^{\alpha_i (X(z_i) + \frac{Q}{2} \ln \hat{g}(z_i))} \prod_{j=1}^M e^{\frac{\beta_j}{2} (X(s_j) + \frac{Q}{2} \ln \hat{g}(s_j))} e^{-S_L(X, \hat{g})}, \quad (2.2.2)$$

for N, M in $\mathbb{N}$. The philosophy of this heuristic definition is the following. Starting from a formal uniform measure $D_{\hat{g}} X$ on the space of maps $\Sigma = \{X : \mathbb{D} \mapsto \mathbb{R}\}$, we add a density given by $e^{-S_L(X,\hat{g})}$. This is simply the Boltzmann weight framework of statistical physics where the probability of a given state (here a map X) is proportional to exponential minus its energy (here the Liouville action). Following [17, 40] it turns out that it is possible to give a rigorous probabilistic definition to (2.2.2) in terms of GMC measures. To do this we will interpret the quantity $e^{-\frac{1}{4\pi} \int_{\mathbb{H}} |\partial^{\hat{g}} X|^2 \hat{g}(z) dz^2} D_{\hat{g}} X$ as the formal density of a GFF in the following sense. We introduce the centered Gaussian field X on $\mathbb{H}$ with covariance given for $x, y \in \mathbb{H}$ by,[11]

$$\mathbb{E}[X(x)X(y)] = \ln \frac{1}{|x-y||x-\overline{y}|} + \ln |x+i|^2 + \ln |y+i|^2 - 2 \ln 2, \quad (2.2.3)$$

and such that (using $K_{\hat{g}} = 1$):

$$\int_{\mathbb{R}} X(s) \hat{g}(s)^{1/2} ds = \int_{\mathbb{R}} K_{\hat{g}}(s) X(s) \hat{g}(s)^{1/2} ds = 0. \quad (2.2.4)$$

Since X lives in the space of distributions we will need again to introduce a cut-off or regularization procedure. For $\delta > 0$ let:

$$\mathbb{H}_\delta = \{z \in \mathbb{H} | \Im(z) > \delta\}.$$

Then let $\rho : [0, +\infty) \mapsto [0, +\infty)$ be a $\mathcal{C}^\infty$ function with compact support in $[0,1]$ and such that $\pi \int_0^\infty \rho(t) dt = 1$. For $x \in \mathbb{H}$ we write $\rho_\epsilon(x) = \frac{1}{\epsilon^2} \rho(\frac{x \overline{x}}{\epsilon^2})$. Then for $z \in \mathbb{H}_\delta$ and $\epsilon < \delta$, we define X_ϵ by

$$X_\epsilon(z) = (\rho_\epsilon * X)(z) = \int_{\mathbb{H}} d^2x X(x) \rho_\epsilon(z-x), \quad (2.2.5)$$

[11] The covariance of the GFF X on $\mathbb{H}$ differs from the covariance (2.1.1) of X on $\mathbb{D}$. The GFF on $\mathbb{D}$ is the image of the GFF on $\mathbb{H}$ by the conformal map $z \mapsto \frac{z-i}{z+i}$ linking $\mathbb{H}$ and $\mathbb{D}$.

and for $s \in \mathbb{R}$ by

$$X_\epsilon(s) = 2(\rho_\epsilon * X)(s) = 2\int_{\mathbb{H}} d^2x X(x)\rho_\epsilon(s-x). \tag{2.2.6}$$

The idea of our regularization is that for a point $z \in \mathbb{H}_\delta$ at a distance at least δ from the boundary and for $\epsilon < \delta$ we can smooth our field $X(z)$ with ρ on a ball of radius ϵ around z. For a point $s \in \mathbb{R}$ we will always write our convolution on the half ball contained in $\mathbb{H}$ of size ϵ. We now define the correlation functions by the following limit,

$$\langle \prod_{i=1}^N V_{\alpha_i}(z_i) \prod_{j=1}^M V_{\beta_j}(s_j)\rangle_{\mathbb{H}} = \lim_{\epsilon\to 0} \langle \prod_{i=1}^N V_{\alpha_i}(z_i) \prod_{j=1}^M V_{\beta_j}(s_j)\rangle_{\mathbb{H},\epsilon}, \tag{2.2.7}$$

where

$$\langle \prod_{i=1}^N V_{\alpha_i}(z_i) \prod_{j=1}^M V_{\beta_j}(s_j)\rangle_{\mathbb{H},\epsilon} = \int_{\mathbb{R}} dce^{-Qc}\mathbb{E}[\prod_{i=1}^N \epsilon^{\frac{\alpha_i^2}{2}} e^{\alpha_i(X_\epsilon(z_i)+\frac{Q}{2}\ln \hat{g}(z_i)+c)} \prod_{j=1}^M \epsilon^{\frac{\beta_j^2}{4}} e^{\frac{\beta_j}{2}(X_\epsilon(s_j)+\frac{Q}{2}\ln\hat{g}(s_j)+c)}$$
$$\times \exp(-\mu_\partial \int_{\mathbb{R}} \epsilon^{\frac{\gamma^2}{4}} e^{\frac{\gamma}{2}(X_\epsilon + c + \frac{Q}{2}\ln \hat{g})} ds]. \tag{2.2.8}$$

The above definition may appear to be convoluted but it is simply the consequence of removing $e^{-\frac{1}{4\pi}\int_{\mathbb{H}} |\partial^{\hat{g}} X|^2 \hat{g}(z) dz^2} D_{\hat{g}}X$ from (2.2.2) and saying that X now becomes $X + c$, where X is our GFF of covariance (2.2.3) and c is a constant integrated with respect to the Lebesgue measure on $\mathbb{R}$. This c is called the zero mode in physics, it corresponds to the fact that $|\partial^{\hat{g}} X|^2$ only determines the field up to a constant (see [17] for more details). To obtain (2.2.8) we have also used (2.2.4) and the explicit values of $R_{\hat{g}}$ and $K_{\hat{g}}$. The limit (2.2.8) exists and is non zero if and only if the insertions obey the Seiberg bounds which are:

$$\sum_{i=1}^N \alpha_i + \frac{1}{2}\sum_{j=1}^M \beta_j > Q \quad \text{and} \quad \forall j, \quad \beta_j < Q. \tag{2.2.9}$$

When the bounds (2.2.9) are satisfied we will write the correlations in the following way:

$$\langle \prod_{i=1}^N V_{\alpha_i}(z_i) \prod_{j=1}^M V_{\beta_j}(s_j)\rangle_{\mathbb{H}} = \int_{\mathbb{R}} dce^{(\sum_i \alpha_i + \sum_j \frac{\beta_j}{2} - Q)c}\mathbb{E}[\prod_{i=1}^N V_{\alpha_i}(z_i) \prod_{j=1}^M V_{\beta_j}(s_j) \exp(-\mu_\partial \int_{\mathbb{R}} e^{\frac{\gamma}{2}(X(s)+\frac{Q}{2}\ln\hat{g}(s)+c)} ds)].$$

We now derive all the formulas that we will need to prove Theorem 2.2.2. In order for the following to work correctly, we need to replace all the $\mathbb{E}[X(x)X(y)]$ by the exact log kernel $\ln \frac{1}{|x-y||x-\overline{y}|}$ or in other words we need to eliminate the dependence on the background metric $\hat{g}$. This will be a consequence of the following identity:

Lemma 2.2.1. *For insertions (z_i, α_i) and (s_j, β_j) satisfying the Seiberg bounds (2.2.9) we have*

$$\frac{\mu_\partial \gamma}{2}\int_{\mathbb{R}} ds \langle V_\gamma(s) \prod_{i=1}^N V_{\alpha_i}(z_i) \prod_{j=1}^M V_{\beta_j}(s_j)\rangle_{\mathbb{H}} = (\sum_{i=1}^N \alpha_i + \sum_{j=1}^M \frac{\beta_j}{2} - Q)\langle \prod_{i=1}^N V_{\alpha_i}(z_i) \prod_{j=1}^M V_{\beta_j}(s_j)\rangle_{\mathbb{H}}.$$

Proof. We perform the change of variable $\frac{2}{\gamma}\ln\mu_\partial + c = c'$ in the following expression:

$$\langle\prod_{i=1}^N V_{\alpha_i}(z_i)\prod_{j=1}^M V_{\beta_j}(s_j)\rangle_{\mathbb{H}} = \int_{\mathbb{R}} dc e^{(\sum_i \alpha_i+\sum_j \frac{\beta_j}{2}-Q)c}\mathbb{E}[\prod_{i=1}^N V_{\alpha_i}(z_i)\prod_{j=1}^M V_{\beta_j}(s_j)\exp(-\mu_\partial\int_{\mathbb{R}} e^{\frac{\gamma}{2}(X(s)+\frac{Q}{2}\ln\hat{g}(s)+c)}ds)]$$

$$= \mu_\partial^{-\frac{2\sum_i\alpha_i+\sum_j\beta_j-2Q}{\gamma}}\int_{\mathbb{R}} dc' e^{(\sum_i \alpha_i+\sum_j \frac{\beta_j}{2}-Q)c'}\mathbb{E}[\prod_{i=1}^N V_{\alpha_i}(z_i)\prod_{j=1}^M V_{\beta_j}(s_j)\exp(-\int_{\mathbb{R}} e^{\frac{\gamma}{2}(X(s)+\frac{Q}{2}\ln\hat{g}(s)+c')}ds)].$$

We then obtain the desired result by differentiating with respect to μ_∂. ☐

So far we have introduced correlation functions of Liouville theory with an arbitrary number of insertions points. For the purposes of Theorem 2.2.2 we will only need to consider correlations with two bulk insertions $z, z_1 \in \mathbb{H}$ of weights $-\frac{\gamma}{2}$ and α and eventually boundary insertions $s, t \in \mathbb{R}$ of weight γ. The value $-\frac{\gamma}{2}$ is called the degenerate weight in the language of CFT. It is for this specific value (and also for the dual weight $-\frac{2}{\gamma}$) that a correlation function containing $V_{-\frac{\gamma}{2}}(z)$ will obey a BPZ differential equation. In the forthcoming computations we will extensively use the shorthand notations:

$$\langle z, z_1\rangle := \langle V_\alpha(z_1)V_{-\frac{\gamma}{2}}(z)\rangle_{\mathbb{H}}, \qquad \langle s, z, z_1\rangle := \langle V_\gamma(s)V_\alpha(z_1)V_{-\frac{\gamma}{2}}(z)\rangle_{\mathbb{H}},$$
$$\langle z, z_1\rangle_\epsilon := \langle V_\alpha(z_1)V_{-\frac{\gamma}{2}}(z)\rangle_{\mathbb{H},\epsilon}, \quad \langle s, z, z_1\rangle_\epsilon := \langle V_\gamma(s)V_\alpha(z_1)V_{-\frac{\gamma}{2}}(z)\rangle_{\mathbb{H},\epsilon}.$$

Our goal is now to compute the derivatives of $\langle V_\alpha(z_1)V_{-\frac{\gamma}{2}}(z)\rangle_{\mathbb{H}}$ in order to prove Theorem 2.2.2. We will illustrate how this computation works with $\partial_z\langle V_\alpha(z_1)V_{-\frac{\gamma}{2}}(z)\rangle_{\mathbb{H}}$. We must use our regularization procedure so we fix $\delta > 0$ and choose $z, z_1 \in \mathbb{H}_\delta$ and $\epsilon < \delta$. We show that $z \mapsto \langle V_\alpha(z_1)V_{-\frac{\gamma}{2}}(z)\rangle_{\mathbb{H}}$ is $\mathcal{C}^1$ on $\mathbb{H}_\delta \setminus \{z_1\}$ $\forall \delta > 0$, which means it is $\mathcal{C}^1$ on $\mathbb{H}\setminus\{z_1\}$. We compute the derivative of the regularized partition function (2.2.8) with respect to z:

$$\begin{aligned}\partial_z\langle z, z_1\rangle_\epsilon = &-\frac{\gamma}{2}\int_{\mathbb{R}} dce^{-Qc}\mathbb{E}[\partial_z(X_\epsilon(z)+\frac{Q}{2}\ln\hat{g}(z))\epsilon^{\frac{\alpha^2}{2}}e^{\alpha(X_\epsilon(z_1)+\frac{Q}{2}\ln\hat{g}(z_1)+c)}\epsilon^{\frac{\gamma^2}{8}}e^{-\frac{\gamma}{2}(X_\epsilon(z)+\frac{Q}{2}\ln\hat{g}(z)+c)}\\ &\times\exp(-\mu_\partial\int_{\mathbb{R}}\epsilon^{\frac{\gamma^2}{4}}e^{\frac{\gamma}{2}(X_\epsilon+c+\frac{Q}{2}\ln\hat{g})}ds]\\ &= -\frac{\gamma}{2}\langle(\partial_z X_\epsilon(z))V_\alpha(z_1)V_{-\frac{\gamma}{2}}(z)\rangle_{\mathbb{H},\epsilon} - \frac{\gamma Q}{4}(\partial_z\ln\hat{g}(z))\langle z, z_1\rangle_{\mathbb{H},\epsilon}\end{aligned}$$

The idea of [46] to compute the first term in the above expression is to realize that we can perform an integration by parts on the underlying Gaussian measure of X.[12] We introduce $X(f) = \int_{\mathbb{H}} X(x)f(x)dx^2$ for some smooth f with compact support and we get:

$$\langle X(f)V_\alpha(z_1)V_{-\frac{\gamma}{2}}(z)\rangle_{\mathbb{H},\epsilon} = \alpha\mathbb{E}[X(f)X_\epsilon(z_1)]\langle z, z_1\rangle_\epsilon - \frac{\gamma}{2}\mathbb{E}[X(f)X_\epsilon(z)]\langle z, z_1\rangle_\epsilon - \mu_\partial\frac{\gamma}{2}\int_{\mathbb{R}}\mathbb{E}[X(f)X_\epsilon(s)]\langle s, z, z_1\rangle_\epsilon ds$$

Since $\partial_z X_\epsilon(z) = \int_{\mathbb{H}} d^2x X(x)\partial_z\rho_\epsilon(z-x)$, we will apply the above formula to $f(x) = \partial_z\rho_\epsilon(z-x)$.

[12] Recall that for a centered Gaussian vector $(X, Y_1, \ldots, Y_N)$ and a smooth function f on $\mathbb{R}^N$, the Gaussian integration by parts yields $\mathbb{E}[Xf(Y_1,\ldots,Y_N)] = \sum_{k=1}^N \mathbb{E}[XY_k]\mathbb{E}[\partial_{Y_k}f(Y_1,\ldots,Y_N)]$. We use the same fact in infinite dimensions on our GFF X.

Using (2.2.3), (2.2.5), (2.2.6), we compute:

$$
\begin{aligned}
\mathbb{E}[\partial_z X_\epsilon(z)X_\epsilon(z_1)] &= \int_{\mathbb{H}}\int_{\mathbb{H}} d^2x d^2y \mathbb{E}[X(x)X(y)]\, \partial_z\rho_\epsilon(z-x)\rho_\epsilon(z_1-y) \\
&= \int_{\mathbb{H}}\int_{\mathbb{H}} d^2x d^2y\, \partial_x \mathbb{E}[X(x)X(y)]\rho_\epsilon(z-x)\rho_\epsilon(z_1-y) \\
&= -\frac{1}{2}\int_{\mathbb{H}}\int_{\mathbb{H}} d^2x d^2y (\frac{1}{x-y}+\frac{1}{x-\overline{y}})\rho_\epsilon(z-x)\rho_\epsilon(z_1-y) - \frac{1}{2}\int_{\mathbb{H}} d^2x\, \partial_x \ln\hat{g}(x)\rho_\epsilon(z-x) \\
\mathbb{E}[\partial_z X_\epsilon(z)X_\epsilon(z)] &= \int_{\mathbb{H}}\int_{\mathbb{H}} d^2x d^2y \mathbb{E}[X(x)X(y)]\, \partial_z\rho_\epsilon(z-x)\rho_\epsilon(z-y) \\
&= \partial_z \frac{1}{2}\int_{\mathbb{H}}\int_{\mathbb{H}} d^2x d^2y \ln\frac{1}{|x-y|}\rho_\epsilon(z-x)\rho_\epsilon(z-y) \\
&+ \int_{\mathbb{H}}\int_{\mathbb{H}} d^2x d^2y\, \partial_x(\ln\frac{1}{|x-\overline{y}|} - \frac{1}{2}\ln\hat{g}(x))\rho_\epsilon(z-x)\rho_\epsilon(z-y) \\
&= -\frac{1}{2}\int_{\mathbb{H}}\int_{\mathbb{H}} d^2x d^2y \frac{1}{x-\overline{y}}\rho_\epsilon(z-x)\rho_\epsilon(z-y) - \frac{1}{2}\int_{\mathbb{H}} d^2x\, \partial_x \ln\hat{g}(x)\rho_\epsilon(z-x) \\
\mathbb{E}[\partial_z X_\epsilon(z)X_\epsilon(s)] &= 2\int_{\mathbb{H}}\int_{\mathbb{H}} d^2x d^2y \mathbb{E}[X(x)X(y)]\, \partial_z\rho_\epsilon(z-x)\rho_\epsilon(s-y) \\
&= -\int_{\mathbb{H}}\int_{\mathbb{H}} d^2x d^2y (\frac{1}{x-y}+\frac{1}{x-\overline{y}})\rho_\epsilon(z-x)\rho_\epsilon(s-y) - \frac{1}{2}\int_{\mathbb{H}} d^2x\, \partial_x \ln\hat{g}(x)\rho_\epsilon(z-x)
\end{aligned}
$$

Putting everything together and taking $\epsilon \to 0$ we arrive at:

$$
\begin{aligned}
\partial_z \langle z_1, z\rangle_{\mathbb{H}} &= (-\frac{\gamma^2}{8}\frac{1}{z-\overline{z}} + \frac{\gamma}{4}\alpha(\frac{1}{z-z_1}+\frac{1}{z-\overline{z_1}}))\langle z, z_1\rangle - \frac{\mu_\partial\gamma^2}{4}\int_{\mathbb{R}}\frac{1}{z-s}\langle s, z, z_1\rangle ds \\
&+ \left(\frac{\gamma}{4}(\alpha-\frac{\gamma}{2}-Q)\langle z, z_1\rangle - \frac{\mu_\partial\gamma^2}{8}\int_{\mathbb{R}}\langle s, z, z_1\rangle ds\right)\partial_z \ln\hat{g}(z) \\
&= (-\frac{\gamma^2}{8}\frac{1}{z-\overline{z}} + \frac{\gamma}{4}\alpha(\frac{1}{z-z_1}+\frac{1}{z-\overline{z_1}}))\langle z, z_1\rangle - \frac{\mu_\partial\gamma^2}{4}\int_{\mathbb{R}}\frac{1}{z-s}\langle s, z, z_1\rangle ds
\end{aligned}
$$

To cancel the metric dependent terms in the last line we have used Lemma 2.2.1. The derivatives ∂_{z_1}, $\partial_{\overline{z}}$, $\partial_{\overline{z_1}}$, and ∂_{zz} are computed along the same lines, their expressions are given in the proof of the theorem below.

2.2.2 The BPZ differential equation

Our goal here is to prove the following result:

Theorem 2.2.2. *Let $\gamma \in (0,2)$ and $\alpha > Q + \frac{\gamma}{2}$. Then $(z_1, z) \mapsto \langle V_{-\frac{\gamma}{2}}(z)V_\alpha(z_1)\rangle_{\mathbb{H}}$ is $\mathcal{C}^2$ on the set $\{z_1, z \in \mathbb{H} | z_1 \neq z\}$ and is solution of the following PDE*

$$
(\frac{4}{\gamma^2}\partial_{zz} + \frac{\Delta_{-\frac{\gamma}{2}}}{(z-\overline{z})^2} + \frac{\Delta_\alpha}{(z-z_1)^2} + \frac{\Delta_\alpha}{(z-\overline{z_1})^2} + \frac{1}{z-\overline{z}}\partial_{\overline{z}} + \frac{1}{z-z_1}\partial_{z_1} + \frac{1}{z-\overline{z_1}}\partial_{\overline{z_1}})\langle V_{-\frac{\gamma}{2}}(z)V_\alpha(z_1)\rangle_{\mathbb{H}} = 0
$$

where $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$, $\Delta_\alpha = \frac{\alpha}{2}(Q - \frac{\alpha}{2})$ and $\Delta_{-\frac{\gamma}{2}} = -\frac{\gamma}{4}(Q + \frac{\gamma}{4})$.

Proof. First let us note that the condition $\alpha > Q + \frac{\gamma}{2}$ corresponds exactly to the Seiberg bounds (2.2.9) and that the $\Delta_\alpha, \Delta_{-\frac{\gamma}{2}}$ are the so-called conformal weights of CFT. Following the method given above we compute all the derivatives that we need:

$$\begin{aligned}
\partial_{z_1}\langle z, z_1\rangle &= (\frac{\alpha\gamma}{4}(\frac{1}{z_1 - z} + \frac{1}{z_1 - \overline{z}}) - \frac{\alpha^2}{2}\frac{1}{z_1 - \overline{z_1}})\langle z, z_1\rangle + \frac{\alpha\mu_\partial\gamma}{2}\int_{\mathbb{R}} \frac{1}{z_1 - s}\langle s, z, z_1\rangle ds \\
\partial_{\overline{z_1}}\langle z, z_1\rangle &= (\frac{\alpha\gamma}{4}(\frac{1}{\overline{z_1} - z} + \frac{1}{\overline{z_1} - \overline{z}}) - \frac{\alpha^2}{2}\frac{1}{\overline{z_1} - z_1})\langle z, z_1\rangle + \frac{\alpha\mu_\partial\gamma}{2}\int_{\mathbb{R}} \frac{1}{\overline{z_1} - s}\langle s, z, z_1\rangle ds \\
\partial_{z}\langle z, z_1\rangle &= (-\frac{\gamma^2}{8}\frac{1}{z - \overline{z}} + \frac{\gamma}{4}\alpha(\frac{1}{z - z_1} + \frac{1}{z - \overline{z_1}}))\langle z, z_1\rangle - \frac{\mu_\partial\gamma^2}{4}\int_{\mathbb{R}} \frac{1}{z - s}\langle s, z, z_1\rangle ds \\
\partial_{\overline{z}}\langle z, z_1\rangle &= (-\frac{\gamma^2}{8}\frac{1}{\overline{z} - z} + \frac{\gamma}{4}\alpha(\frac{1}{\overline{z} - z_1} + \frac{1}{\overline{z} - \overline{z_1}}))\langle z, z_1\rangle - \frac{\mu_\partial\gamma^2}{4}\int_{\mathbb{R}} \frac{1}{\overline{z} - s}\langle s, z, z_1\rangle ds
\end{aligned}$$

$$\begin{aligned}
\partial_{zz}\langle z, z_1\rangle = &(\frac{\gamma^2}{8}\frac{1}{(z - \overline{z})^2} - \frac{\gamma}{4}\alpha(\frac{1}{(z - z_1)^2} + \frac{1}{(z - \overline{z_1})^2}))\langle z, z_1\rangle \\
&+ (\frac{\gamma^2}{8}\frac{1}{z - \overline{z}} - \frac{\gamma\alpha}{4}(\frac{1}{z - z_1} + \frac{1}{z - \overline{z_1}}))^2\langle z, z_1\rangle + \frac{\mu_\partial\gamma^2}{4}\int_{\mathbb{R}} \frac{1}{(z - s)^2}\langle s, z, z_1\rangle ds \\
&- \frac{\mu_\partial\gamma^3}{8}(-\frac{\gamma}{2}\frac{1}{z - \overline{z}} + \alpha(\frac{1}{z - z_1} + \frac{1}{z - \overline{z_1}}))\int_{\mathbb{R}} \frac{1}{z - s}\langle s, z, z_1\rangle ds \\
&+ \frac{\gamma^4\mu_\partial^2}{16}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{z - s}\frac{1}{z - t}\langle s, t, z, z_1\rangle dsdt - \frac{\mu_\partial\gamma^4}{16}\int_{\mathbb{R}} \frac{1}{(z - s)^2}\langle s, z, z_1\rangle ds. \qquad (2.2.10)
\end{aligned}$$

Again the shorthand notation $\langle s, t, z, z_1\rangle$ stands for $\langle V_\gamma(s)V_\gamma(t)V_\alpha(z_1)V_{-\frac{\gamma}{2}}(z)\rangle_{\mathbb{H}}$ with $z, z_1 \in \mathbb{H}$ and $s, t \in \mathbb{R}$. We start by checking that all the terms without μ_∂ cancel correctly, we gather them based on their α-dependence. Terms with α^2:

$$\alpha^2\left(-\frac{1}{4(z - z_1)^2} - \frac{1}{4(z - \overline{z_1})^2} + \frac{1}{4}(\frac{1}{z - z_1} + \frac{1}{z - \overline{z_1}})^2 - \frac{1}{2(z - z_1)(z_1 - \overline{z_1})} + \frac{1}{2(z - \overline{z_1})(z_1 - \overline{z_1})}\right)\langle z, z_1\rangle = 0.$$

Terms with α:

$$\begin{aligned}
&\alpha\left((\frac{Q}{2} - \frac{\gamma}{4} - \frac{1}{\gamma})(\frac{1}{(z - z_1)^2} + \frac{1}{(z - \overline{z_1})^2}) - \frac{\gamma}{4}\frac{1}{z - \overline{z}}(\frac{1}{z - z_1} + \frac{1}{z - \overline{z_1}})\right)\langle z, z_1\rangle \\
&+\alpha\left(\frac{\gamma}{4}(\frac{1}{z - z_1}\frac{1}{z_1 - \overline{z}} + \frac{1}{z - \overline{z_1}}\frac{1}{\overline{z_1} - \overline{z}}) - \frac{\gamma}{4}\frac{1}{z - \overline{z}}(\frac{1}{z_1 - \overline{z}} + \frac{1}{\overline{z_1} - \overline{z}})\right)\langle z, z_1\rangle = 0.
\end{aligned}$$

Terms with no α:

$$(\frac{1}{2} + \frac{\gamma^2}{8} + \frac{\gamma^2}{16} + \Delta_{-\frac{\gamma}{2}})\frac{1}{(z - \overline{z})^2}\langle z, z_1\rangle = 0.$$

We must now make sure that all the terms with μ_∂ cancel correctly, for this we need to perform an integration by parts. However there is a slight subtlety coming from the fact that the derivative ∂_s applied to $\langle s, z, z_1\rangle$ gives a term in $\frac{1}{s-t}\langle s, t, z, z_1\rangle$ and $\frac{1}{s-t}$ is not integrable. But this difficulty can be easily overcome with our regularization procedure. We get,

$$\frac{\mu_\partial\gamma^2}{4}\int_{\mathbb{R}}\frac{1}{(z-s)^2}\langle s,z,z_1\rangle_\epsilon ds=\frac{\mu_\partial\gamma^2}{4}\int_{\mathbb{R}}\partial_s\frac{1}{(z-s)}\langle s,z,z_1\rangle_\epsilon ds=-\frac{\mu_\partial\gamma^2}{4}\int_{\mathbb{R}}\frac{1}{(z-s)}\partial_s\langle s,z,z_1\rangle_\epsilon ds$$
$$\underset{\epsilon\to0}{\longrightarrow}-\frac{\mu_\partial\gamma^2}{4}\int_{\mathbb{R}}\frac{1}{(z-s)}(\frac{\gamma^2}{4}(\frac{1}{(s-z)}+\frac{1}{(s-\overline{z})})-\frac{\gamma}{2}\alpha(\frac{1}{(s-z_1)}+\frac{1}{(s-\overline{z_1})}))\langle s,z,z_1\rangle ds$$
$$-\frac{\mu_\partial^2\gamma^4}{8}\lim_{\epsilon\to0}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{1}{(z-s)}\frac{1}{(s-t)_{\epsilon,\epsilon}}\langle s,t,z,z_1\rangle_\epsilon dtds,$$

where we have introduced

$$\frac{1}{(s-t)_{\epsilon,\epsilon}}=4\int_{\mathbb{H}}\int_{\mathbb{H}}d^2x_1d^2x_2\frac{1}{s+x_1-t-x_2}\rho_\epsilon(x_1)\rho_\epsilon(x_2).$$

We symmetrize the last term:

$$-\frac{\mu_\partial^2\gamma^4}{8}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{1}{(z-s)}\frac{1}{(s-t)_{\epsilon,\epsilon}}\langle s,t,z,z_1\rangle_\epsilon dtds$$
$$=-\frac{\mu_\partial^2\gamma^4}{16}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{1}{(s-t)_{\epsilon,\epsilon}}(\frac{1}{(z-s)}-\frac{1}{(z-t)})\langle s,t,z,z_1\rangle_\epsilon dtds$$
$$=-\frac{\mu_\partial^2\gamma^4}{16}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{1}{(z-s)(z-t)}\frac{s-t}{(s-t)_{\epsilon,\epsilon}}\langle s,t,z,z_1\rangle_\epsilon dtds$$
$$\underset{\epsilon\to0}{\longrightarrow}-\frac{\gamma^4\mu_\partial^2}{16}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{1}{z-s}\frac{1}{z-t}\langle s,t,z,z_1\rangle dtds. \tag{2.2.11}$$

From the above we see that the double terms in $\int_{\mathbb{R}}\int_{\mathbb{R}}$ coming from (2.2.10) and (2.2.11) cancel correctly. Finally we look at the cross terms:

$$\mu_\partial\frac{\gamma^2}{4}\int_{\mathbb{R}}(\frac{1}{z-s}\frac{1}{\overline{z}-s}+\frac{1}{z-\overline{z}}\frac{1}{z-s}-\frac{1}{\overline{z}-s}\frac{1}{z-\overline{z}})\langle s,z,z_1\rangle ds=0$$
$$\mu_\partial\frac{\alpha\gamma}{2}\int_{\mathbb{R}}(\frac{1}{z-s}\frac{1}{s-z_1}-\frac{1}{z-z_1}\frac{1}{z-s}+\frac{1}{z-z_1}\frac{1}{z_1-s})\langle s,z,z_1\rangle ds=0$$
$$\mu_\partial\frac{\alpha\gamma}{2}\int_{\mathbb{R}}(\frac{1}{z-s}\frac{1}{s-\overline{z}_1}-\frac{1}{z-\overline{z}_1}\frac{1}{z-s}+\frac{1}{z-\overline{z}_1}\frac{1}{\overline{z}_1-s})\langle s,z,z_1\rangle ds=0,$$

and therefore we have proved Theorem 2.2.2. □

2.2.3 Correlation functions as moments of GMC on the unit circle

We are now going to express our correlation function $\langle V_{-\frac{\gamma}{2}}(z)V_\alpha(z_1)\rangle_{\mathbb{H}}$ as a moment of Gaussian multiplicative chaos (GMC) on the unit circle and turn the BPZ equation of Theorem 2.2.2 into a differential equation on $G(\gamma,p,t)$. As explained in the appendix, the KPZ relation of [40] tells us that we have the realtion

$$\langle V_\alpha(z_1)V_{-\frac{\gamma}{2}}(z)\rangle_{\mathbb{H}}=\frac{1}{|z_1-\overline{z}_1|^{2\Delta_\alpha-2\Delta_{-\frac{\gamma}{2}}}}\frac{1}{|z-\overline{z}_1|^{4\Delta_{-\frac{\gamma}{2}}}}\langle V_\alpha(0)V_{-\frac{\gamma}{2}}(t)\rangle_{\mathbb{D}} \tag{2.2.12}$$

where Δ_α and $\Delta_{-\frac{\gamma}{2}}$ are defined as in Theorem 2.2.2 and where $\langle V_{-\frac{\gamma}{2}}(t)V_\alpha(0)\rangle_{\mathbb{D}}$ is the correlation of LCFT defined on the unit disk. From the results of [40] we can express it in terms of inverse moments of the GMC measure on the unit circle,

$$\begin{aligned}\langle V_{-\frac{\gamma}{2}}(t)V_\alpha(0)\rangle_{\mathbb{D}} &= \frac{2}{\gamma}\mu_\partial^{-\frac{2\alpha-\gamma-2Q}{\gamma}}\Gamma(\frac{2\alpha-\gamma-2Q}{\gamma})t^{\frac{\alpha\gamma}{2}}(1-t^2)^{-\frac{\gamma^2}{8}}\mathbb{E}\left[(\int_{\partial\mathbb{D}} e^{\frac{\gamma}{2}(X(e^{i\theta})-2\alpha\ln|e^{i\theta}|+\gamma\ln|e^{i\theta}-t|)}d\theta)^{-\frac{2\alpha-\gamma-2Q}{\gamma}}\right]\\ &= \frac{2}{\gamma}\mu_\partial^{-\frac{2\alpha-\gamma-2Q}{\gamma}}\Gamma(\frac{2\alpha-\gamma-2Q}{\gamma})t^{\frac{\alpha\gamma}{2}}(1-t^2)^{-\frac{\gamma^2}{8}}G(\gamma,p,t), \end{aligned} \quad (2.2.13)$$

where we have the following relation between our parameters p and α:

$$-p = \frac{2\alpha-\gamma-2Q}{\gamma}. \quad (2.2.14)$$

The condition $\alpha > Q+\frac{\gamma}{2}$ of Theorem 2.2.2 is then precisely the condition $p < 0$ of Proposition 2.1.2. It is then a long but straightforward computation to turn the BPZ equation on $\langle V_\alpha(z_1)V_{-\frac{\gamma}{2}}(z)\rangle_{\mathbb{H}}$ into the differential equation on $G(\gamma,p,t)$, more details on this can be found in the appendix. Therefore we have proved Proposition 2.1.2.

Let us make a few comments on our method. One might attempt to prove Proposition 2.1.2 without introducing at all the framework of Liouville theory on $\mathbb{H}$ and by just computing partial derivatives directly on the function $G(\gamma,p,t)$. In this computation all terms cancel easily except a few terms coming from the second derivative in t for which it is very difficult to see that they equal zero. It appears that seeing that all terms cancel correctly by performing the computation directly on the circle is just as complicated as proving Theorem 2.1.1. This is due to the fact that fractional moments of GMC are very hard to manipulate. Liouville theory seems to be the correct framework where the computations are tractable and thus the KPZ relation below (2.2.12) - a highly non trivial change of variable - is a key ingredient of our proof. On the other hand there is great hope to adapt our method to obtain more exact formulas on GMC measures in other cases, for instance on the unit interval $[0,1]$ or on the two-dimensional GMC measure on the unit disk $\mathbb{D}$.

2.3 The shift equation for $U(\gamma,p)$

The goal of this section is to identify the coefficients C_1, C_2, B_1 and B_2 of Proposition 2.1.3 to find a link between $U(\gamma,p)$ and $U(\gamma,p-1)$. The result we expect to find is:

$$U(\gamma,p) = \frac{2\pi\Gamma(1-p\frac{\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})\Gamma(1-(p-1)\frac{\gamma^2}{4})}U(\gamma,p-1). \quad (2.3.1)$$

We will perform asymptotic expansions of $G(\gamma,p,t)$ in $t\to 0$ and $t\to 1$ to obtain the desired result.

2.3.1 Asymptotic expansion in $t\to 0$

Since $t\mapsto G(\gamma,p,t)$ is a continuous function on $[0,1]$, we have

$$G(\gamma,p,0) = U(\gamma,p)$$

and since $p<0$, we cannot have a term in $t^{\frac{\gamma^2}{2}(p-1)}$ in the expression $G(\gamma,p,t)$ and therefore we get:

$$C_2 = 0. \tag{2.3.2}$$

Then taking $t=0$ in the expression of $G(\gamma,p,t)$, we find:

$$C_1 = U(\gamma,p). \tag{2.3.3}$$

2.3.2 Asymptotic expansion in $t \to 1$

Taking $t=1$ in the expression of $G(\gamma,p,t)$, we get:

$$B_1 = \mathbb{E}[(\int_0^{2\pi} |1-e^{i\theta}|^{\frac{\gamma^2}{2}} e^{\frac{\gamma}{2}X(e^{i\theta})} d\theta)^p]. \tag{2.3.4}$$

At this stage there is nothing we can do with this coefficient but as an output of our proof we will also obtain a value for this quantity, see Corollary 2.1.5. We must now go to the next order to find B_2. We introduce the notation $h_u(t) = |t-e^{iu}|^{\frac{\gamma^2}{2}}$. In the following computations we will extensively use the Girsanov theorem (also called the Cameron-Martin formula) in the following way:

$$\mathbb{E}[\int_0^{2\pi} e^{\frac{\gamma}{2}X(e^{iu})} du (\int_0^{2\pi} e^{\frac{\gamma}{2}X(e^{i\theta})} d\theta)^{p-1}] = \int_0^{2\pi} \mathbb{E}[(\int_0^{2\pi} \frac{e^{\frac{\gamma}{2}X(e^{i\theta})}}{|e^{iu}-e^{i\theta}|^{\frac{\gamma^2}{2}}} d\theta)^{p-1}] du. \tag{2.3.5}$$

We then write:

$$\begin{aligned} &\mathbb{E}[(\int_0^{2\pi} |t-e^{i\theta}|^{\frac{\gamma^2}{2}} e^{\frac{\gamma}{2}X(e^{i\theta})} d\theta)^p] - \mathbb{E}[(\int_0^{2\pi} |1-e^{i\theta}|^{\frac{\gamma^2}{2}} e^{\frac{\gamma}{2}X(e^{i\theta})} d\theta)^p] \\ &= p\mathbb{E}[\int_0^{2\pi} du (|t-e^{iu}|^{\frac{\gamma^2}{2}} - |1-e^{iu}|^{\frac{\gamma^2}{2}}) e^{\frac{\gamma}{2}X(e^{iu})} (\int_0^{2\pi} |1-e^{i\theta}|^{\frac{\gamma^2}{2}} e^{\frac{\gamma}{2}X(e^{i\theta})} d\theta)^{p-1}] + R(t) \\ &= p\int_0^{2\pi} du (h_u(t)-h_u(1)) \mathbb{E}[(\int_0^{2\pi} \frac{|1-e^{i\theta}|^{\frac{\gamma^2}{2}}}{|e^{iu}-e^{i\theta}|^{\frac{\gamma^2}{2}}} e^{\frac{\gamma}{2}X(e^{i\theta})} d\theta)^{p-1}] + R(t). \end{aligned} \tag{2.3.6}$$

$R(t)$ are higher order terms that are given by the Taylor formula applied to $x \mapsto x^p$:

$$\begin{aligned} R(t) &= p(p-1) \int_0^{2\pi}\int_0^{2\pi} d\theta_1 d\theta_2 (h_{\theta_1}(t)-h_{\theta_1}(1))(h_{\theta_2}(t)-h_{\theta_2}(1)) \\ &\times \mathbb{E}\left[\int_{s=0}^1 ds(1-s) \left(\int_0^{2\pi} d\theta \frac{h_\theta(1)+s(h_\theta(t)-h_\theta(1))}{|e^{i\theta_1}-e^{i\theta}|^{\frac{\gamma^2}{2}} |e^{i\theta_2}-e^{i\theta}|^{\frac{\gamma^2}{2}}} e^{\frac{\gamma}{2}X(e^{i\theta})}\right)^{p-2}\right]. \end{aligned}$$

One may wonder if the GMC measures with fractional powers that appear in the above computations are well-defined. The answer is that for $\beta \in \mathbb{R}$, $\int_0^{2\pi} \frac{1}{|1-e^{i\theta}|^{\frac{\beta\gamma}{2}}} e^{X(e^{i\theta})} d\theta < +\infty \; a.s. \Leftrightarrow \beta < \frac{\gamma}{2} + \frac{2}{\gamma}$. Therefore in the expression of $R(t)$ the only problem is when $\theta_1 = \theta_2$. But in our case we are dealing with negative moments so at $\theta_1 = \theta_2$ we simply get 0. Now the following integral coming from

(2.3.6) can also be expressed in terms of hypergeometric functions F:

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}(h_u(t)-h_u(1))du &= F(-\frac{\gamma^2}{4},-\frac{\gamma^2}{4},1,t^2)-F(-\frac{\gamma^2}{4},-\frac{\gamma^2}{4},1,1) \\ &= \frac{\Gamma(\frac{\gamma^2}{2}+1)}{\Gamma(\frac{\gamma^2}{4}+1)^2}F(-\frac{\gamma^2}{4},-\frac{\gamma^2}{4},-\frac{\gamma^2}{2},1-t^2)-F(-\frac{\gamma^2}{4},-\frac{\gamma^2}{4},1,1) \\ &+\frac{\Gamma(-\frac{\gamma^2}{2}-1)}{\Gamma(-\frac{\gamma^2}{4})^2}(1-t^2)^{1+\frac{\gamma^2}{2}}F(1+\frac{\gamma^2}{4},1+\frac{\gamma^2}{4},2+\frac{\gamma^2}{2},1-t^2).\end{aligned} \tag{2.3.7}$$

In the last line we have used the formula (2.4.10) given in the appendix valid here for $\gamma \neq \sqrt{2}$. We first look at the case where $0<\gamma<\sqrt{2}$. We notice that in this case $1<1+\frac{\gamma^2}{2}<2$ and that $u\mapsto h'_u(1)$ is integrable in $u=0$ but not $u\mapsto h''_u(1)$. (2.3.7) tells us that:

$$\lim_{t\to 1}\frac{1}{(1-t^2)^{1+\frac{\gamma^2}{2}}}\frac{1}{2\pi}\int_0^{2\pi}(h_u(t)-h_u(1)-(t-1)h'_u(1))du=\frac{\Gamma(-\frac{\gamma^2}{2}-1)}{\Gamma(-\frac{\gamma^2}{4})^2}. \tag{2.3.8}$$

The key observation is that the same result holds if we add some continuous function c defined on the unit circle:

Lemma 2.3.1. *For $\epsilon>0$ and $u\in[0,2\pi]$ let $h_u(1-\epsilon)=|1-\epsilon-e^{iu}|^{\frac{\gamma^2}{2}}$ and let $c:\partial\mathbb{D}\mapsto\mathbb{R}$ be a continuous function defined on the unit circle. Then we have:*

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon^{1+\frac{\gamma^2}{2}}}\frac{1}{2^{1+\frac{\gamma^2}{2}}}\frac{1}{2\pi}\int_0^{2\pi}(h_u(1-\epsilon)-h_u(1)+\epsilon h'_u(1))c(e^{iu})du=\frac{\Gamma(-\frac{\gamma^2}{2}-1)}{\Gamma(-\frac{\gamma^2}{4})^2}c(1). \tag{2.3.9}$$

Proof. We start by showing that

$$\frac{1}{\epsilon^{1+\frac{\gamma^2}{2}}}\int_0^{2\pi}|h_u(1-\epsilon)-h_u(1)+\epsilon h'_u(1)|du$$

remains bounded as ϵ goes to 0. We split the integral into two parts, $\int_{-\epsilon}^{\epsilon}$ and $\int_{\epsilon}^{2\pi-\epsilon}$. To analyse the first part we can perform an asymptotic expansion on $u\mapsto e^{iu}$, we get:

$$\begin{aligned}&\frac{1}{\epsilon^{1+\frac{\gamma^2}{2}}}\int_{-\epsilon}^{\epsilon}|h_u(1-\epsilon)-h_u(1)+\epsilon h'_u(1)|du \\ &\leq \frac{M_1}{\epsilon^{1+\frac{\gamma^2}{2}}}\int_{-\epsilon}^{\epsilon}||\epsilon+iu|^{\frac{\gamma^2}{2}}-|u|^{\frac{\gamma^2}{2}}+\epsilon\frac{\gamma^2}{4}|u|^{\frac{\gamma^2}{2}}|du \\ &= \frac{M_1}{\epsilon^{1+\frac{\gamma^2}{2}}}\int_{-1}^{1}||\epsilon+i\epsilon v|^{\frac{\gamma^2}{2}}-|\epsilon v|^{\frac{\gamma^2}{2}}+\epsilon\frac{\gamma^2}{4}|\epsilon v|^{\frac{\gamma^2}{2}}|\epsilon dv\leq M_2\end{aligned}$$

for some $M_1,M_2>0$. The other part of the integral can be bounded by the Taylor formula:

$$\begin{aligned}\frac{1}{\epsilon^{1+\frac{\gamma^2}{2}}}\int_{\epsilon}^{2\pi-\epsilon}|h_u(1-\epsilon)-h_u(1)+\epsilon h'_u(1)|du &\leq \frac{1}{\epsilon^{1+\frac{\gamma^2}{2}}}\int_{\epsilon}^{2\pi-\epsilon}\frac{\epsilon^2}{2}\sup_{x\in[1-\epsilon,1]}|h''_u(x)|du \\ &\leq M_3\epsilon^{1-\frac{\gamma^2}{2}}\int_{\epsilon}^{1}|u|^{\frac{\gamma^2}{2}-2}du\leq M_4\end{aligned}$$

for some constants $M_3, M_4 > 0$. By continuity of c, for $\epsilon' > 0$ there exists an $\eta > 0$ such that $\forall u \in (-\eta, \eta)$, $|c(e^{iu}) - c(1)| \leq \epsilon'$. We can then write:

$$\begin{aligned}
&\frac{1}{\epsilon^{1+\frac{\gamma^2}{2}}} \int_0^{2\pi} |h_u(1-\epsilon) - h_u(1) + \epsilon h_u'(1)||c(e^{iu}) - c(1)| du \\
&= \frac{1}{\epsilon^{1+\frac{\gamma^2}{2}}} \int_{-\eta}^{\eta} |h_u(1-\epsilon) - h_u(1) + \epsilon h_u'(1)||c(e^{iu}) - c(1)| du \\
&+ \frac{1}{\epsilon^{1+\frac{\gamma^2}{2}}} \int_{\eta}^{2\pi-\eta} |h_u(1-\epsilon) - h_u(1) + \epsilon h_u'(1)||c(e^{iu}) - c(1)| du \\
&\leq \tilde{M}_1 \epsilon' + \frac{1}{\epsilon^{1+\frac{\gamma^2}{2}}} \epsilon^2 \tilde{M}_2
\end{aligned}$$

for some $\tilde{M}_1, \tilde{M}_2 > 0$. Since the above is true for all ϵ' we easily arrive at:

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^{1+\frac{\gamma^2}{2}}} \int_0^{2\pi} |h_u(1-\epsilon) - h_u(1) + \epsilon h_u'(1)||c(e^{iu}) - c(1)| du = 0.$$

From this and using the exact computation (2.3.8) we obtain (2.3.9). □

We then apply the Lemma 2.3.1 to our problem by choosing:

$$c(e^{iu}) = \mathbb{E}[(\int_0^{2\pi} \frac{|1-e^{i\theta}|^{\frac{\gamma^2}{2}}}{|e^{iu} - e^{i\theta}|^{\frac{\gamma^2}{2}}} e^{\frac{\gamma}{2} X(e^{i\theta})} d\theta)^{p-1}].$$

Finally we bounded the higher order terms $R(t)$ by:

$$\begin{aligned}
|R(t)| &\leq M \int_0^{2\pi} \int_0^{2\pi} d\theta_1 d\theta_2 (h_{\theta_1}(t) - h_{\theta_1}(1))(h_{\theta_2}(t) - h_{\theta_2}(1)) \\
&\leq \tilde{M}(1-t)^2
\end{aligned}$$

for some $M, \tilde{M} > 0$. Combining all the above arguments we find the following expansion in powers of $(1-t)$ for $G(\gamma, p, t)$:

$$G(\gamma,p,t) = B_1 + p(t-1)\int_0^{2\pi} du h_u'(1) \mathbb{E}[(\int_0^{2\pi} d\theta \frac{|1-e^{i\theta}|^{\frac{\gamma^2}{2}} e^{\frac{\gamma}{2}X(e^{i\theta})}}{|e^{iu}-e^{i\theta}|^{\frac{\gamma^2}{2}}})^{p-1}] + 2\pi p \frac{\Gamma(-\frac{\gamma^2}{2}-1)}{\Gamma(-\frac{\gamma^2}{4})^2} U(\gamma, p-1)(1-t^2)^{1+\frac{\gamma^2}{2}} + o((1-t)^{1+\frac{\gamma^2}{2}}).$$

This gives the value of the coefficient B_2:

$$B_2 = 2\pi p \frac{\Gamma(-\frac{\gamma^2}{2}-1)}{\Gamma(-\frac{\gamma^2}{4})^2} U(\gamma, p-1). \tag{2.3.10}$$

In the case $\sqrt{2} < \gamma < 2$ we need to go one order further in the computations. We now have that $u \to h_u''(1)$ is integrable in $u = 0$. If we go one order further in (2.3.8) we still get:

$$\lim_{t \to 1} \frac{1}{(1-t^2)^{1+\frac{\gamma^2}{2}}} \frac{1}{2\pi} \int_0^{2\pi} (h_u(t) - h_u(1) - (t-1)h_u'(1) - \frac{1}{2}(t-1)^2 h_u''(1)) du = \frac{\Gamma(-\frac{\gamma^2}{2}-1)}{\Gamma(-\frac{\gamma^2}{4})^2}. \tag{2.3.11}$$

Lemma 2.3.1 still holds if we go one order further and the analysis of $R(t)$ gives this time:

$$R(t) = m(1-t)^2 + O((1-t)^3) \tag{2.3.12}$$

for some $m \in \mathbb{R}$. Indeed since we can write

$$h_{\theta_1}(t) - h_{\theta_1}(1) = (t-1)h'_{\theta_1}(1) + \frac{(t-1)^2}{2}h''_{\theta_1}(1) + o((t-1)^2),$$

and we have

$$R(t) = p(p-1)(t-1)^2 \int_0^{2\pi}\int_0^{2\pi} h'_{\theta_1}(1)h'_{\theta_2}(1)\mathbb{E}\left[\int_{s=0}^1 ds(1-s)\left(\int_0^{2\pi} d\theta \frac{h_\theta(1)+s(h_\theta(t)-h_\theta(1))}{|e^{i\theta_1}-e^{i\theta}|^{\frac{\gamma^2}{2}}|e^{i\theta_2}-e^{i\theta}|^{\frac{\gamma^2}{2}}} e^{\frac{\gamma}{2}X(e^{i\theta})}\right)^{p-2}\right] + O((t-1)^3),$$

applying one last Taylor expansion on the above expectation $\mathbb{E}[.]$ we get

$$\mathbb{E}\left[\int_{s=0}^1 ds(1-s)\left(\int_0^{2\pi} d\theta \frac{h_\theta(1)+s(h_\theta(t)-h_\theta(1))}{|e^{i\theta_1}-e^{i\theta}|^{\frac{\gamma^2}{2}}|e^{i\theta_2}-e^{i\theta}|^{\frac{\gamma^2}{2}}} e^{\frac{\gamma}{2}X(e^{i\theta})}\right)^{p-2}\right] = \frac{1}{2}\mathbb{E}\left[\left(\int_0^{2\pi} d\theta \frac{h_\theta(1)e^{\frac{\gamma}{2}X(e^{i\theta})}}{|e^{i\theta_1}-e^{i\theta}|^{\frac{\gamma^2}{2}}|e^{i\theta_2}-e^{i\theta}|^{\frac{\gamma^2}{2}}}\right)^{p-2}\right] + O(t-1)$$

and so we finally arrive at (2.3.12). From the above we see that we can write an expansion of $G(\gamma,p,t)$ of the form:

$$G(\gamma,p,t) = B_1 + b_1(1-t) + b_2(1-t)^2 + 2\pi p\frac{\Gamma(-\frac{\gamma^2}{2}-1)}{\Gamma(-\frac{\gamma^2}{4})^2}U(\gamma,p-1)(1-t^2)^{1+\frac{\gamma^2}{2}} + o((1-t)^{1+\frac{\gamma^2}{2}}).$$

for some $b_1, b_2 \in \mathbb{R}$. From this we deduce (2.3.10) in the case $\sqrt{2} < \gamma < 2$.[13]

To conclude we have identified explicitly C_1, C_2, and B_2. Using the change of basis formula (2.4.9) and considering that $C_2 = 0$, the relationship between the other two coefficients is:

$$B_2 = \frac{\Gamma(-1-\frac{\gamma^2}{2})\Gamma(\frac{\gamma^2}{4}(1-p)+1)}{\Gamma(-\frac{\gamma^2}{4})\Gamma(-\frac{\gamma^2 p}{4})}C_1. \tag{2.3.13}$$

Finally we arrive at the relation for $p < 0$:

$$\frac{U(\gamma,p)}{U(\gamma,p-1)} = \frac{2\pi p\Gamma(-\frac{\gamma^2 p}{4})}{\Gamma(-\frac{\gamma^2}{4})\Gamma(1-(p-1)\frac{\gamma^2}{4})} = \frac{2\pi\Gamma(1-p\frac{\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})\Gamma(1-(p-1)\frac{\gamma^2}{4})}. \tag{2.3.14}$$

By continuity of $p \mapsto U(\gamma,p)$ we can take the limit $p \to 0$ in the above relation to get the shift equation for all $p \leq 0$. Therefore we have proved Proposition 2.1.4.

2.4 Appendix

2.4.1 Mapping of BPZ to the unit disk

In this section we turn the BPZ equation we found on $\mathbb{H}$ using Liouville theory into an equation on the function $G(\gamma,p,t)$. We recall that the conformal map $\psi_1 : x \mapsto \frac{x-i}{x+i}$ maps the upper half plane $\mathbb{H}$ equipped with metric $\hat{g}(x) = \frac{4}{|x+i|^4}$ to the unit disk $\mathbb{D}$ equipped with the Euclidean metric. The KPZ formula of [40] for a change of domain then tells us that:

$$\langle V_\alpha(z_1)V_{-\frac{\gamma}{2}}(z)\rangle_{\mathbb{H}} = |\psi_1'(z_1)|^{2\Delta_\alpha}|\psi_1'(z)|^{2\Delta_{-\frac{\gamma}{2}}}\langle V_\alpha(\psi_1(z_1))V_{-\frac{\gamma}{2}}(\psi_1(z))\rangle_{\mathbb{D}}. \tag{2.4.1}$$

[13] The case $\gamma = \sqrt{2}$ is left out here as $\Gamma(-\frac{\gamma^2}{2}-1)$ in (2.3.10) is ill-defined but we can solve this problem by using the continuity of $\gamma \mapsto U(\gamma,p)$, a simple exercise, and by taking the limit $\gamma \to \sqrt{2}$ in (2.3.14).

We apply again the KPZ formula of [40] to the conformal map ψ_2 of $\mathbb{D}$ onto $\mathbb{D}$ that maps $\psi_1(z_1)$ to 0 and $\psi_1(z)$ to $t \in (0,1)$. More explicitly $\psi_2(x) = e^{i\theta}\frac{x-\psi_1(z_1)}{1-x\overline{\psi_1(z_1)}}$ with θ chosen such that $\psi_2(\psi_1(z)) \in (0,1)$. This gives:

$$\langle V_\alpha(\psi_1(z_1))V_{-\frac{\gamma}{2}}(\psi_1(z))\rangle_{\mathbb{D}} = |\psi_2'(\psi_1(z_1))|^{2\Delta_\alpha}|\psi_2'(\psi_1(z))|^{2\Delta_{-\frac{\gamma}{2}}}\langle V_\alpha(0)V_{-\frac{\gamma}{2}}(t)\rangle_{\mathbb{D}}. \tag{2.4.2}$$

Combining both of the above relations we arrive at:

$$\langle V_\alpha(z_1)V_{-\frac{\gamma}{2}}(z)\rangle_{\mathbb{H}} = \frac{1}{|z_1-\overline{z}_1|^{2\Delta_\alpha - 2\Delta_{-\frac{\gamma}{2}}}}\frac{1}{|z-\overline{z}_1|^{4\Delta_{-\frac{\gamma}{2}}}}\langle V_\alpha(0)V_{-\frac{\gamma}{2}}(t)\rangle_{\mathbb{D}}. \tag{2.4.3}$$

Now starting from our BPZ equation,

$$(\frac{4}{\gamma^2}\partial_{zz} + \frac{\Delta_{-\frac{\gamma}{2}}}{(z-\overline{z})^2} + \frac{\Delta_\alpha}{(z-z_1)^2} + \frac{\Delta_\alpha}{(z-\overline{z}_1)^2} + \frac{1}{z-\overline{z}}\partial_{\overline{z}} + \frac{1}{z-z_1}\partial_{z_1} + \frac{1}{z-\overline{z_1}}\partial_{\overline{z_1}})\langle V_\alpha(z_1)V_{-\frac{\gamma}{2}}(z)\rangle_{\mathbb{H}} = 0,$$

we obtain the following equation for $\langle V_\alpha(0)V_{-\frac{\gamma}{2}}(t)\rangle_{\mathbb{D}}$:

$$\left(\frac{t^2}{\gamma^2}\frac{d^2}{dt^2} + (-\frac{t}{\gamma^2} + \frac{2t^3-t}{2(1-t^2)})\frac{d}{dt} + (\Delta_\alpha + \Delta_{-\frac{\gamma}{2}}\frac{2t^2-t^4}{(t^2-1)^2})\right)\langle V_\alpha(0)V_{-\frac{\gamma}{2}}(t)\rangle_{\mathbb{D}} = 0. \tag{2.4.4}$$

Then using (2.2.13) and (2.2.14), we get the announced differential equation of Proposition 2.1.2 for $G(\gamma,p,t)$,

$$(t(1-t^2)\frac{\partial^2}{\partial t^2} + (t^2-1)\frac{\partial}{\partial t} + 2(C-(A+B+1)t^2)\frac{\partial}{\partial t} - 4ABt)G(\gamma,p,t) = 0, \tag{2.4.5}$$

with $A = -\frac{\gamma^2 p}{4}$, $B = -\frac{\gamma^2}{4}$, and $C = \frac{\gamma^2}{4}(1-p)+1$.

2.4.2 Hypergeometric functions

We recall here briefly all the fact on hypergeometric functions that we have used throughout our paper. For A, B, C, and x real numbers we introduce the following power series,

$$F(A,B,C,x) = \sum_{n=0}^{\infty}\frac{A_n B_n}{n! C_n}x^n, \tag{2.4.6}$$

where we have set for $n \in \mathbb{N}$, $A_n = \frac{\Gamma(A+n)}{\Gamma(A)}$, Γ being the standard gamma function. The function F is the hypergeometric series and it can be used to solve the following hypergeometric equation:

$$(x(1-x)\frac{\partial^2}{\partial x^2} + (C-(A+B+1)x)\frac{\partial}{\partial x} - AB)H(x) = 0. \tag{2.4.7}$$

The solutions of this equation can be given in two different bases, one corresponding to an expansion in powers of x and the other in powers of $1-x$. We write:

$$H(x) = C_1 F(A,B,C,x) + C_2 x^{1-C}F(1+A-C,1+B-C,2-C,x), \tag{2.4.8}$$
$$H(x) = B_1 F(A,B,1+A+B-C,1-x) + B_2(1-x)^{C-A-B}F(C-A,C-B,1+C-A-B,1-x).$$

In our case where $A = -\frac{\gamma^2 p}{4}$, $B = -\frac{\gamma^2}{4}$, and $C = \frac{\gamma^2}{4}(1-p)+1$, the four real constants C_1, C_2, B_1, B_2 are linked by the following relation:

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} \frac{\Gamma(1+\frac{\gamma^2}{2})\Gamma(\frac{\gamma^2}{4}(1-p)+1)}{\Gamma(1+\frac{\gamma^2}{4})\Gamma(\frac{\gamma^2}{4}(2-p)+1)} & \frac{\Gamma(1+\frac{\gamma^2}{2})\Gamma(\frac{\gamma^2}{4}(p-1)+1)}{\Gamma(1+\frac{\gamma^2}{4})\Gamma(\frac{\gamma^2}{4}p+1)} \\ \frac{\Gamma(-1-\frac{\gamma^2}{2})\Gamma(\frac{\gamma^2}{4}(1-p)+1)}{\Gamma(-\frac{\gamma^2}{4})\Gamma(-\frac{\gamma^2 p}{4})} & \frac{\Gamma(-1-\frac{\gamma^2}{2})\Gamma(\frac{\gamma^2}{4}(p-1)+1)}{\Gamma(-\frac{\gamma^2}{4})\Gamma(\frac{\gamma^2}{4}(p-2))} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}. \tag{2.4.9}$$

This relation can easily be deduced from the following properties of the hypergeometric function:

$$\begin{aligned} F(A,B,C,x) = {} & \frac{\Gamma(C)\Gamma(C-A-B)}{\Gamma(C-A)\Gamma(C-B)} F(A,B,A+B-C+1,1-x) \\ & + (1-x)^{C-A-B} \frac{\Gamma(C)\Gamma(A+B-C)}{\Gamma(A)\Gamma(B)} F(C-A,C-B,C-A-B+1,1-x) \end{aligned} \tag{2.4.10}$$

and

$$F(A,B,C,1) = \frac{\Gamma(C)\Gamma(C-A-B)}{\Gamma(C-A)\Gamma(C-B)}. \tag{2.4.11}$$

More details on hypergeometric functions can be found in [3].

Chapter 3

Gaussian multiplicative chaos on the unit interval

In collaboration with Tunan Zhu.

We consider a sub-critical Gaussian multiplicative chaos (GMC) measure defined on the unit interval $[0, 1]$ and prove an exact formula for the fractional moments of the total mass of this measure. Our formula includes the case where log-singularities (also called insertion points) are added in 0 and 1, the most general case predicted by the Selberg integral. The idea to perform this computation is to introduce certain auxiliary functions resembling holomorphic observables of conformal field theory that will be solution of hypergeometric equations. Solving these equations then provides non-trivial relations that completely determine the moments we wish to compute. We also include a detailed discussion of the so-called reflection coefficients appearing in tail expansions of GMC measures and Liouville theory. Our theorem provides an exact value for one of these coefficients. Lastly we mention some additional applications to the maximum of the log-correlated field on the interval and to random hermitian matrices.

3.1 Introduction and main result

Starting from a log-correlated field X one can define by standard regularization techniques the associated Gaussian multiplicative chaos (GMC). It corresponds to a random measure whose density with respect to the Lebesgue measure is formally given by the exponential of X. This definition is formal as X lives in the space of distributions but since the pioneering work of Kahane [42] in 1985 it is well understood how to give a rigorous probabilistic definition to these GMC measures by using a limiting procedure. Ever since GMC has been extensively studied in probability theory and mathematical physics with applications including 3d turbulence, statistical physics, mathematical finance, random geometry and 2d quantum gravity. See for instance [72] for a review.

Despite the importance of GMC measures in many active fields of research, rigorous computations have remained until very recently completely out of reach. A large number of formulas have been conjectured by the physicists' trick of analytic continuation from positive integers to real numbers (see explanations below) but with no indication of how to prove such formulas. A decisive step was made in [17] where a connection is uncovered between GMC measures and the correlation functions of Liouville conformal field theory (LCFT). By implementing the techniques of conformal field theory (CFT) in a probabilistic setting one can hope to perform exact computations on GMC.

Indeed, in 2017 a proof was given by Kupiainen-Rhodes-Vargas of the celebrated DOZZ formula [46, 47] first conjectured independently by Dorn and Otto in [22] and by Zamolodchikov and Zamolodchikov in [86]. This formula gives the value of the three-point correlation function of LCFT on the Riemann sphere and it can also be seen as the first exact computation of fractional moments of a GMC measure. Very shortly after, the study of LCFT on the unit disk by the first author lead in [70] to the proof of an exact probability density for the total mass of the GMC measure on the unit circle. This result proves the conjecture of Fyodorov and Bouchaud stated in [31] and is the first probability density for a GMC measure.

The present chapter presents a third case where exact computations are tractable using CFT-inspired techniques which is the case of GMC on the unit interval $[0,1]$ with X of covariance written below (3.1.1). This model was studied by Bacry-Muzy in [7] where they prove existence of moments and other properties of GMC. Five years after exact formulas for this model on the interval were conjectured independently by Fyodorov et al. in [35, 34] and by Ostrovsky, see [60] and references therein. In [35, 34] the exact formulas are found using an analytic continuation from integers to real numbers but in his papers Ostrovsky went a step further and showed that the formulas did correspond to a valid probability distribution. He also performs the computation of the derivatives of all order in γ of (3.1.4) at $\gamma = 0$ which is referred to as the intermittency differentiation. However a crucial analycity argument is missing for this approach to prove rigorously an exact formula. See [63] for a beautiful review on all the known results and conjectures for the GMC on the interval (and also for the similar model on the circle) as well as for many additional references. The main result of our work is precisely the proof of these conjectures for the GMC measure on $[0,1]$.

Let us now introduce the framework of our paper. We consider the log-correlated field X on the interval $[0,1]$ with covariance given for $x, y \in [0,1]$ by:

$$\mathbb{E}[X(x)X(y)] = 2\ln\frac{1}{|x-y|}.\text{[1]} \tag{3.1.1}$$

We define the associated GMC measure on the interval $[0,1]$ by the standard regularization procedure for $\gamma \in (0,2)$,

$$e^{\frac{\gamma}{2}X(x)}dx := \lim_{\epsilon\to 0} e^{\frac{\gamma}{2}X_\epsilon(x)-\frac{\gamma^2}{8}\mathbb{E}[X_\epsilon(x)^2]}dx, \tag{3.1.2}$$

where X_ϵ stands for any reasonable cut-off of X that converges to X as ϵ goes to 0. The convergence in (3.1.2) is in probability with respect to the weak topology of measures, meaning that for all continuous test functions $f : [0,1] \mapsto \mathbb{R}$ the following holds in probability:

$$\int_0^1 f(x)e^{\frac{\gamma}{2}X(x)}dx = \lim_{\epsilon\to 0}\int_0^1 f(x)e^{\frac{\gamma}{2}X_\epsilon(x)-\frac{\gamma^2}{8}\mathbb{E}[X_\epsilon(x)^2]}dx. \tag{3.1.3}$$

For an elementary proof of this convergence see [9]. We now introduce the main quantity of interest of our paper, for $\gamma \in (0,2)$ and for real p, a, b:

$$M(\gamma,p,a,b) := \mathbb{E}[(\int_0^1 x^a(1-x)^b e^{\frac{\gamma}{2}X(x)}dx)^p]. \tag{3.1.4}$$

This quantity is the moment p of the total mass of our GMC measure with two "insertion points" in 0 and 1 of weight a and b. The theory of Gaussian multiplicative chaos tells us that these

[1] Our normalization differs from the $\ln\frac{1}{|x-y|}$ usually found in the literature.

moments are non-trivial, i.e. different from 0 and $+\infty$, if and only if:

$$a > -\frac{\gamma^2}{4} - 1, \quad b > -\frac{\gamma^2}{4} - 1, \quad p < \frac{4}{\gamma^2} \wedge (1 + \frac{4}{\gamma^2}(1+a)) \wedge (1 + \frac{4}{\gamma^2}(1+b)). \tag{3.1.5}$$

The first two conditions are required for the GMC measure to integrate the fractional powers x^a and $(1-x)^b$. Notice that this condition is weaker than the one we would get with Lebesgue measure, $a > -1$ and $b > -1$.[2] We then have a bound on the moment p, the first part $p < \frac{4}{\gamma^2}$ is the standard condition for the existence of a moment of GMC without insertions. The additional condition on p, $p < (1 + \frac{4}{\gamma^2}(1+a)) \wedge (1 + \frac{4}{\gamma^2}(1+b))$, comes from the presence of the insertions. A proof of the bounds (3.1.5) can be found in [73, 40].

Now the goal of our work is simply to prove the following exact formula for $M(\gamma, p, a, b)$:

Theorem 3.1.1. *For $\gamma \in (0,2)$ and for p, a, b satisfying (3.1.5)[3], $M(\gamma, p, a, b)$ is given by,*

$$\frac{(2\pi)^p (\frac{2}{\gamma})^{p\frac{\gamma^2}{4}}}{\Gamma(1-\frac{\gamma^2}{4})^p} \frac{\Gamma_\gamma(\frac{2}{\gamma}(a+1) - (p-1)\frac{\gamma}{2})\Gamma_\gamma(\frac{2}{\gamma}(b+1) - (p-1)\frac{\gamma}{2})\Gamma_\gamma(\frac{2}{\gamma}(a+b+2) - (p-2)\frac{\gamma}{2})\Gamma_\gamma(\frac{2}{\gamma} - p\frac{\gamma}{2})}{\Gamma_\gamma(\frac{2}{\gamma})\Gamma_\gamma(\frac{2}{\gamma}(a+1) + \frac{\gamma}{2})\Gamma_\gamma(\frac{2}{\gamma}(b+1) + \frac{\gamma}{2})\Gamma_\gamma(\frac{2}{\gamma}(a+b+2) - (2p-2)\frac{\gamma}{2})},$$

where the function $\Gamma_\gamma(x)$ is defined for $x > 0$ and $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$ by:

$$\ln \Gamma_\gamma(x) = \int_0^\infty \frac{dt}{t} \left[\frac{e^{-xt} - e^{-\frac{Qt}{2}}}{(1 - e^{-\frac{\gamma t}{2}})(1 - e^{-\frac{2t}{\gamma}})} - \frac{(\frac{Q}{2} - x)^2}{2} e^{-t} + \frac{x - \frac{Q}{2}}{t} \right]. \tag{3.1.6}$$

As a corollary by choosing $a = b = 0$ we obtain the value of the moments of the GMC measure without insertions:

Corollary 3.1.2. *For $\gamma \in (0,2)$ and $p < \frac{4}{\gamma^2}$:*

$$\mathbb{E}[(\int_0^1 e^{\frac{\gamma}{2}X(x)} dx)^p] = \frac{(2\pi)^p (\frac{2}{\gamma})^{p\frac{\gamma^2}{4}}}{\Gamma(1-\frac{\gamma^2}{4})^p} \frac{\Gamma_\gamma(\frac{2}{\gamma} - (p-1)\frac{\gamma}{2})^2 \Gamma_\gamma(\frac{4}{\gamma} - (p-2)\frac{\gamma}{2})\Gamma_\gamma(\frac{2}{\gamma} - p\frac{\gamma}{2})}{\Gamma_\gamma(\frac{2}{\gamma})\Gamma_\gamma(\frac{2}{\gamma} + \frac{\gamma}{2})^2 \Gamma_\gamma(\frac{4}{\gamma} - (2p-2)\frac{\gamma}{2})}. \tag{3.1.7}$$

Thanks to the computations performed by Ostrovsky [61], we can also state our main result in the following equivalent way:

Corollary 3.1.3. *The following equality in law holds,*

$$\int_0^1 x^a (1-x)^b e^{\frac{\gamma}{2}X(x)} dx = \frac{2\pi 2^{-(3(1+\frac{\gamma^2}{4}) + 2(a+b))}}{\Gamma(1-\frac{\gamma^2}{4})} L X_1 X_2 X_3 Y, \tag{3.1.8}$$

where L, X_1, X_2, X_3, Y are five independent random variables in $\mathbb{R}_+$ with the following laws. L is a log-normal law, more precisely the exponential of $\mathcal{N}(0, \gamma^2 \ln 2)$. Y has a probability density with

[2] Proving Theorem 3.1.1 for $-1 - \frac{\gamma^2}{4} < a \leq -1$ will require a lot of technical work as precise estimates on GMC measures are required to show that Proposition 3.1.4 holds in this case.

[3] The result also holds for all complex p such that $\mathrm{Re}(p)$ satisfies the bounds (3.1.5).

respect to Lebesgue given by $\frac{4}{\gamma^2} y^{-1-\frac{4}{\gamma^2}} \exp(-y^{-\frac{4}{\gamma^2}}) \mathbf{1}_{[0,+\infty)}(y)$. *The variable* X_1, X_2, X_3 *each follow the inverse of a special* $\beta_{2,2}$ *law defined in appendix 3.4.5:*

$$X_1 = \beta_{2,2}^{-1}(1, \frac{4}{\gamma^2}; 1 + \frac{4}{\gamma^2}(1+a), \frac{2(b-a)}{\gamma^2}, \frac{2(b-a)}{\gamma^2}),$$
$$X_2 = \beta_{2,2}^{-1}(1, \frac{4}{\gamma^2}; 1 + \frac{2}{\gamma^2}(2+a+b), \frac{1}{2}, \frac{2}{\gamma^2}),$$
$$X_3 = \beta_{2,2}^{-1}(1, \frac{4}{\gamma^2}; 1 + \frac{4}{\gamma^2}, \frac{1}{2} + \frac{2}{\gamma^2}(1+a+b), \frac{1}{2} + \frac{2}{\gamma^2}(1+a+b)).$$

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3.1.1 Strategy of the proof

We start off with the well known observation that a formula can be given for $M(\gamma, p, a, b)$ in the very special case where $p \in \mathbb{N}$, $a > -1$, $b > -1$ and p satisfying (3.1.5). Indeed, in this case the computation reduces to a real integral - the famous Selberg integral - whose value is known, see for instance [30]. This is because for a positive integer moment we can write p integrals and exchange them with the expectation $\mathbb{E}[\cdot]$. More precisely for $a, b > -1$, p satisfying (3.1.5) and $p \in \mathbb{N}$ we have, using any suitable regularization procedure:

$$\begin{aligned}
\mathbb{E}[(\int_0^1 x^a (1-x)^b e^{\frac{\gamma}{2} X(x)} dx)^p] &= \lim_{\epsilon \to 0} \mathbb{E}[(\int_0^1 x^a (1-x)^b e^{\frac{\gamma}{2} X_\epsilon(x) - \frac{\gamma^2}{8} \mathbb{E}[X_\epsilon(x)^2]} dx)^p] \\
&= \lim_{\epsilon \to 0} \int_{[0,1]^p} \prod_{i=1}^p x_i^a (1-x_i)^b \mathbb{E}[\prod_{i=1}^p e^{\frac{\gamma}{2} X_\epsilon(x_i) - \frac{\gamma^2}{8} \mathbb{E}[X_\epsilon(x_i)^2]}] dx_1 \dots dx_p \\
&= \int_{[0,1]^p} \prod_{i=1}^p x_i^a (1-x_i)^b e^{\frac{\gamma^2}{4} \sum_{i<j} \mathbb{E}[X(x_i) X(x_j)]} dx_1 \dots dx_p \\
&= \int_{[0,1]^p} \prod_{i=1}^p x_i^a (1-x_i)^b \prod_{i<j} \frac{1}{|x_i - x_j|^{\frac{\gamma^2}{2}}} dx_1 \dots dx_p \\
&= \prod_{j=1}^p \frac{\Gamma(1 + a - (j-1)\frac{\gamma^2}{4}) \Gamma(1 + b - (j-1)\frac{\gamma^2}{4}) \Gamma(1 - j\frac{\gamma^2}{4})}{\Gamma(2 + a + b - (p+j-2)\frac{\gamma^2}{4}) \Gamma(1 - \frac{\gamma^2}{4})}. \quad (3.1.9)
\end{aligned}$$

The last line is precisely given by the Selberg integral. It is then natural to look for an analytic continuation of this expression from integer to real p satisfying (3.1.5). Notice that giving the analytic continuation of a such a quantity is a highly non-trivial problem as p appears in the argument of the Gamma functions as well as in a number of terms in the product. To find the right candidate for the analytic continuation we start by writing down the following relations that we will refer to as the shift equations. They are deduced by simple algebra from (3.1.9) again for

$p \in \mathbb{N}$ and under the bounds (3.1.5),

$$\frac{M(\gamma,p,a+\frac{\gamma^2}{4},b)}{M(\gamma,p,a,b)} = \frac{\Gamma(1+a+\frac{\gamma^2}{4})\Gamma(2+a+b-(2p-2)\frac{\gamma^2}{4})}{\Gamma(1+a-(p-1)\frac{\gamma^2}{4})\Gamma(2+a+b-(p-2)\frac{\gamma^2}{4})}, \tag{3.1.10}$$

$$\frac{M(\gamma,p,a+1,b)}{M(\gamma,p,a,b)} = \frac{\Gamma(\frac{4}{\gamma^2}(1+a)+1)\Gamma(\frac{4}{\gamma^2}(2+a+b)-(2p-2))}{\Gamma(\frac{4}{\gamma^2}(1+a)-(p-1))\Gamma(\frac{4}{\gamma^2}(2+a+b)-(p-2))}, \tag{3.1.11}$$

and for $p \in \mathbb{N}^*$ under the bounds (3.1.5),

$$\frac{M(\gamma,p,a,b)}{M(\gamma,p-1,a,b)} = \frac{\Gamma(1+a-(p-1)\frac{\gamma^2}{4})\Gamma(1+b-(p-1)\frac{\gamma^2}{4})\Gamma(1-p\frac{\gamma^2}{4})\Gamma(2+a+b-(p-2)\frac{\gamma^2}{4})}{\Gamma(2+a+b-(2p-3)\frac{\gamma^2}{4})\Gamma(2+a+b-(2p-2)\frac{\gamma^2}{4})\Gamma(1-\frac{\gamma^2}{4})}. \tag{3.1.12}$$

Of course similar shift equations hold for b but as there is a symmetry $M(\gamma,p,a,b) = M(\gamma,p,b,a)$ we will write everything only for a. The reason why the function $\Gamma_\gamma(x)$ introduced in Theorem 3.1.1 appears is that it verifies the following two relations, for $\gamma \in (0,2)$ and $x > 0$,

$$\frac{\Gamma_\gamma(x)}{\Gamma_\gamma(x+\frac{\gamma}{2})} = \frac{1}{\sqrt{2\pi}}\Gamma(\frac{\gamma x}{2})(\frac{\gamma}{2})^{-\frac{\gamma x}{2}+\frac{1}{2}}, \tag{3.1.13}$$

$$\frac{\Gamma_\gamma(x)}{\Gamma_\gamma(x+\frac{2}{\gamma})} = \frac{1}{\sqrt{2\pi}}\Gamma(\frac{2x}{\gamma})(\frac{\gamma}{2})^{\frac{2x}{\gamma}-\frac{1}{2}}. \tag{3.1.14}$$

See appendix 3.4.5 for more details on $\Gamma_\gamma(x)$. Therefore we can use $\Gamma_\gamma(x)$ to construct a candidate function that will verify all the shift equations (3.1.10), (3.1.11), (3.1.12) not only for $p \in \mathbb{N}$ but for any real p satisfying the bounds (3.1.5). More precisely for any function $C: p \mapsto C(p)$ of p the following quantity,

$$C(p)\frac{\Gamma_\gamma(\frac{2}{\gamma}(a+1)-(p-1)\frac{\gamma}{2})\Gamma_\gamma(\frac{2}{\gamma}(b+1)-(p-1)\frac{\gamma}{2})\Gamma_\gamma(\frac{2}{\gamma}(a+b+2)-(p-2)\frac{\gamma}{2})}{\Gamma_\gamma(\frac{2}{\gamma}(a+1)+\frac{\gamma}{2})\Gamma_\gamma(\frac{2}{\gamma}(b+1)+\frac{\gamma}{2})\Gamma_\gamma(\frac{2}{\gamma}(a+b+2)-(2p-2)\frac{\gamma}{2})}, \tag{3.1.15}$$

is a solution to the shift equations (3.1.10), (3.1.11). Notice that for $\frac{\gamma^2}{4} \notin \mathbb{Q}$ these two equations completely determine the dependence on a (and on b by symmetry) of $M(\gamma,p,a,b)$ and then by a standard continuity argument in γ we can extend the result to all $\gamma \in (0,2)$. Next the equation (3.1.12) translates into a constraint on the unknown function $C(p)$:

$$\frac{C(p)}{C(p-1)} = \sqrt{2\pi}(\frac{\gamma}{2})^{(p-1)\frac{\gamma^2}{4}-\frac{1}{2}}\frac{\Gamma(1-p\frac{\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})}. \tag{3.1.16}$$

We see that (3.1.16) is not enough to fully determine the function $C(p)$. An additional shift equation that is a priori not predicted by the Selberg integral (3.1.9) is required. We will indeed prove that we have,

$$\frac{C(p)}{C(p-\frac{4}{\gamma^2})} = f(\gamma)(\frac{\gamma}{2})^{-p}\Gamma(\frac{4}{\gamma^2}-p), \tag{3.1.17}$$

where $f(\gamma)$ is an unknown positive function of γ. Now combining (3.1.16) and (3.1.17) completely determines the fonction $C(p)$ again up to an unknown constant c_γ of γ:

$$C(p) = c_\gamma\frac{(2\pi)^p}{\Gamma(1-\frac{\gamma^2}{4})^p}(\frac{2}{\gamma})^{p\frac{\gamma^2}{4}}\Gamma_\gamma(\frac{2}{\gamma}-p\frac{\gamma}{2}). \tag{3.1.18}$$

This last constant c_γ is evaluated by choosing $p=0$ and thus we arrive at the function of Theorem 3.1.1 giving the expression of $M(\gamma,p,a,b)$.

Now the major difficulty that must be overcome is to find a way to prove all the shift equations (3.1.10), (3.1.11), (3.1.12) as well as the additional equation (3.1.17) for all values of p,a,b satisfying (3.1.5) and not just for integer p. This is precisely what is done in section 3.2 where Proposition 3.2.1 completely determines the dependence in a and b of $M(\gamma,p,a,b)$ and Proposition 3.2.2 establishes (3.1.18).

The key ingredient of our proof is to introduce the following two auxiliary functions for $t \leq 0$,

$$U(t) := U(\gamma,p;a,b;t) = \mathbb{E}[(\int_0^1 (x-t)^{\frac{\gamma^2}{4}} x^a (1-x)^b e^{\frac{\gamma}{2}X(x)} dx)^p], \tag{3.1.19}$$

and

$$\tilde{U}(t) := \tilde{U}(\gamma,p;a,b;t) = \mathbb{E}[(\int_0^1 (x-t)\, x^a (1-x)^b \, e^{\frac{\gamma}{2}X(x)} dx)^p], \tag{3.1.20}$$

and to show using probabilistic techniques that the following holds:

Proposition 3.1.4. *Let $0<\gamma<2$, a,b,p satisfy the bounds (3.1.5), then U is solution of the hypergeometric equation for $t<0$:*

$$t(1-t)U''(t) + (C-(A+B+1)t)U'(t) - ABU(t) = 0. \tag{3.1.21}$$

The parameters are given by:

$$A = -\frac{p\gamma^2}{4}, B = -(a+b+1)-(2-p)\frac{\gamma^2}{4}, C = -a-\frac{\gamma^2}{4}. \tag{3.1.22}$$

$\tilde{U}$ is also solution of the hypergeometric equation with different parameters:

$$\tilde{A} = -p, \tilde{B} = -\frac{4}{\gamma^2}(a+b+2)+p-1, \tilde{C} = -\frac{4}{\gamma^2}(a+1). \tag{3.1.23}$$

The auxiliary functions are very similar to the correlation functions of LCFT with a degenerate field insertion - see [46, 47] for the case of the sphere and [70] for the unit disk - which also obey differential equations known as the BPZ equations. What is mysterious in our present case is that it is not clear whether there exists an actual CFT where $U(t)$ and $\tilde{U}(t)$ correspond to correlations with degenerate insertions which would explain why the differential equations of Proposition 3.1.4 hold. Furthermore if we replace the real t by a complex variable $t \in \mathbb{C}\backslash[0,\infty]$, it is not hard to see that $U(t)$ is a holomorphic function and Proposition 3.1.4 will hold if we replace the ordinary derivative by a complex derivative ∂_t. In the conformal bootstrap approach of CFT initiated by Belavin-Polyakov-Zamolodchikov in [8], a correlation function with a degenerate insertion can be decomposed into combinations of the structure constants and conformal blocks. A conformal block is a locally holomorphic function and it is always accompanied by its complex conjugate in the decomposition. What is mysterious with $U(t)$ and $\tilde{U}(t)$ is that we only see the holomorphic part. At this stage we have no CFT-based explanation for this observation although a possible path could be to look at boundary LCFT with multiple boundary cosmological constants, see for instance [57]. On the other hand let us mention that again in the very special case where $p \in \mathbb{N}$, $U(t)$ and $\tilde{U}(t)$ reduce to Selberg-type integrals and the equations of Proposition 3.1.4 were known in this case, see [43].

Once Proposition 3.1.4 is established, the last part of the proof is then to write the solutions of the hypergeometric equations in two different bases. One corresponds to a power series expansion in $|t|$ and the other to an expansion in $|t|^{-1}$. The change of basis formula (3.4.51) written in appendix 3.4.5 given by the theory of hypergeometric functions then provides non-trivial relations which are precisely the shift equations that we wish to prove. Putting everything together we have thus shown Theorem 3.1.1.

3.1.2 Tail expansion for GMC and the reflection coefficients

We include here a very general discussion about tail expansions of GMC with an insertion point and of the corresponding reflection coefficients in dimensions one and two. The first case that was studied is the tail expansion of a GMC in dimension two and a precise asymptotic was given in [47] in terms of the reflection coefficient $\overline{R}_2(\alpha)$,[4] see Proposition 3.1.6 below.[5] Let us also mention that it was recently discovered in [82] that $\overline{R}_2(\alpha)$ corresponds to the partition function of the α-quantum sphere of Duplantier-Miller-Sheffield [26]. Now our exact formula on the unit interval will allow us to write a similar tail expansion for GMC in dimension one. Following [26] we use the standard radial decomposition of the covariance (3.1.1) of X around the point 0, i.e. we write for $s \geq 0$,

$$X(e^{-s/2}) = B_s + Y(e^{-s/2}), \tag{3.1.24}$$

where B_s is a standard Brownian motion and Y is an independent Gaussian process that can be defined on the whole plane with covariance given for $x, y \in \mathbb{C}$ by:

$$\mathbb{E}[Y(x)Y(y)] = 2 \ln \frac{|x| \vee |y|}{|x-y|}. \tag{3.1.25}$$

Motivated by the Williams decomposition of Theorem 3.4.3, we introduce for $\lambda > 0$ the process that will be used in the definitions below,

$$\mathcal{B}_s^{\lambda} := \begin{cases} \hat{B}_s - \lambda s & s \geq 0 \\ \bar{B}_{-s} + \lambda s & s < 0, \end{cases} \tag{3.1.26}$$

where $(\hat{B}_s - \lambda s)_{s \geq 0}$ and $(\bar{B}_s - \lambda s)_{s \geq 0}$ are two independent Brownian motions with negative drift conditioned to stay negative. We can now give the definitions of the two coefficients in dimension one $\overline{R}_1^{\partial}(\alpha)$ and $\overline{R}_1(\alpha)$ along with the associated GMC measures with insertion $I_{1,\eta}^{\partial}(\alpha)$ and $I_{1,\eta}(\alpha)$ whose tail behaviour will be governed by the corresponding coefficient:

$$\overline{R}_1^{\partial}(\alpha) := \mathbb{E}[(\frac{1}{2}\int_{-\infty}^{\infty} e^{\frac{\gamma}{2}\mathcal{B}_s^{\frac{Q-\alpha}{2}}} e^{\frac{\gamma}{2}Y(e^{-s/2})} ds)^{\frac{2}{\gamma}(Q-\alpha)}], \qquad I_{1,\eta}^{\partial}(\alpha) := \int_0^{\eta} x^{-\frac{\gamma\alpha}{2}} e^{\frac{\gamma}{2}X(x)} dx$$

$$\overline{R}_1(\alpha) := \mathbb{E}[(\frac{1}{2}\int_{-\infty}^{\infty} e^{\frac{\gamma}{2}\mathcal{B}_s^{\frac{Q-\alpha}{2}}} (e^{\frac{\gamma}{2}Y(e^{-s/2})} + e^{\frac{\gamma}{2}Y(-e^{-s/2})}) ds)^{\frac{2}{\gamma}(Q-\alpha)}], \quad I_{1,\eta}(\alpha) := \int_{v-\eta}^{v+\eta} |x-v|^{-\frac{\gamma\alpha}{2}} e^{\frac{\gamma}{2}X(x)} dx$$

Let us make some comments on these definitions. Here $\alpha \in (\frac{\gamma}{2}, Q)$, $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$, and η is an arbitrary positive real number chosen small enough. To match the conventions of the study of LCFT we

[4] In [47] or [82] this coefficient is actually called $\overline{R}(\alpha)$ but for the needs of our discussion we introduce the 2 to indicate the dimension. Furthermore the bar stands for the fact that it is the unit volume coefficient.

[5] $\overline{R}_2(\alpha)$ is the bulk reflection coefficient in dimension two, a boundary reflection coefficient $\overline{R}_2^{\partial}(\alpha)$ also exists but its value remains unknown, see the figure below.

have written the fractional power $x^{-\frac{\gamma\alpha}{2}}$, so in these notations we have $a = -\frac{\gamma\alpha}{2}$. Notice that the difference between $I^{\partial}_{1,\eta}(\alpha)$ and $I_{1,\eta}(\alpha)$ lies in the position of the insertion. For $I^{\partial}_{1,\eta}(\alpha)$ the insertion is placed in 0 (by symmetry we could have placed it in 1). Our Theorem 3.1.1 will give us the value of the associated coefficient $\overline{R}^{\partial}_1(\alpha)$. The other case corresponds to placing the insertion at a point v inside the interval, $v \in (0,1)$, and gives the quantity $I_{1,\eta}(\alpha)$. The computation of the associated $\overline{R}_1(\alpha)$ will be done in a future work. We now claim:

Proposition 3.1.5. *For $\alpha \in (\frac{\gamma}{2}, Q)$ we have the following tail expansion for $I^{\partial}_{1,\eta}(\alpha)$ as $u \to \infty$ and for some $\nu > 0$,*

$$\mathbb{P}(I^{\partial}_{1,\eta}(\alpha) > u) = \frac{\overline{R}^{\partial}_1(\alpha)}{u^{\frac{2}{\gamma}(Q-\alpha)}} + O(\frac{1}{u^{\frac{2}{\gamma}(Q-\alpha)+\nu}}), \tag{3.1.27}$$

where the value of $\overline{R}^{\partial}_1(\alpha)$ is given by:

$$\overline{R}^{\partial}_1(\alpha) = \frac{(2\pi)^{\frac{2}{\gamma}(Q-\alpha)-\frac{1}{2}}(\frac{2}{\gamma})^{\frac{\gamma}{2}(Q-\alpha)-\frac{1}{2}}}{(Q-\alpha)\Gamma(1-\frac{\gamma^2}{4})^{\frac{2}{\gamma}(Q-\alpha)}} \frac{\Gamma_\gamma(\alpha-\frac{\gamma}{2})}{\Gamma_\gamma(Q-\alpha)}. \tag{3.1.28}$$

The proof of this proposition is done in appendix 3.4.4. Notice that we impose the condition $\alpha \in (\frac{\gamma}{2}, Q)$. This is crucial for the tail behaviour of $I^{\partial}_{1,\eta}(\alpha)$ (or similarly for $I_{1,\eta}(\alpha)$) to be dominated by the insertion and this is precisely why the asymptotic expansion is independent of the choice of η. It also explains why the radial decomposition (3.1.24) is natural as it is well suited to study X around a particular point. If one is interested in the case where $\alpha < \frac{\gamma}{2}$ (or simply $\alpha = 0$), a different argument known as the localization trick is required to obtain the tail expansion, see [74] for more details. For the sake of completeness of our discussion we also recall the tail expansion in dimension two that was obtained in [47]. The normalizations in this case are slightly different as we do not include a factor 2 in the covariance. We work with a Gaussian process $\tilde{X}$ defined on the unit disk $\mathbb{D}$ with covariance $\ln \frac{1}{|x-y|}$. Instead of Y we use $\tilde{Y}$ with covariance:

$$\mathbb{E}[\tilde{Y}(x)\tilde{Y}(y)] = \ln \frac{|x| \vee |y|}{|x-y|}. \tag{3.1.29}$$

For an insertion placed in z, $|z| < 1$ we now define,

$$\overline{R}_2(\alpha) := \mathbb{E}[(\int_{-\infty}^{\infty} e^{\gamma \mathcal{B}^{Q-\alpha}_s} \int_0^{2\pi} e^{\gamma \tilde{Y}(e^{-s}e^{i\theta})} ds)^{\frac{2}{\gamma}(Q-\alpha)}], \quad I_{2,\eta}(\alpha) := \int_{B(z,\eta)} |x-z|^{-\gamma\alpha} e^{\gamma \tilde{X}(x)} dx^2,$$

and we state the result obtained in [47]:

Proposition 3.1.6. *(Kupiainen-Rhodes-Vargas [47]) For $\alpha \in (\frac{\gamma}{2}, Q)$ we have the following tail expansion for $I_{2,\eta}(\alpha)$ as $u \to \infty$ and for some $\nu > 0$,*

$$\mathbb{P}(I_{2,\eta}(\alpha) > u) = \frac{\overline{R}_2(\alpha)}{u^{\frac{2}{\gamma}(Q-\alpha)}} + O(\frac{1}{u^{\frac{2}{\gamma}(Q-\alpha)+\nu}}), \tag{3.1.30}$$

where the value of $\overline{R}_2(\alpha)$ is given by:

$$\overline{R}_2(\alpha) = -\frac{\gamma}{2(Q-\alpha)} \frac{(\pi\Gamma(\frac{\gamma^2}{4}))^{\frac{2}{\gamma}(Q-\alpha)}}{\Gamma(1-\frac{\gamma^2}{4})^{\frac{2}{\gamma}(Q-\alpha)}} \frac{\Gamma(-\frac{\gamma}{2}(Q-\alpha))}{\Gamma(\frac{\gamma}{2}(Q-\alpha))\Gamma(\frac{2}{\gamma}(Q-\alpha))}. \tag{3.1.31}$$

A similar proposition is also expected for $\overline{R}_2^{\partial}(\alpha)$, the boundary reflection coefficient in dimension two, whose expression and computation is left for a future paper. One notices that $\overline{R}_1^{\partial}(\alpha)$ has a more convoluted expression than $\overline{R}_2(\alpha)$ as the special function Γ_γ appears in its expression. Such expressions have already appeared in the study of Liouville theory for instance in [67] where a general formula for the reflection amplitude is given. We now summarize the four different cases that we have discussed in the following figure. For each coefficient the number 1 or 2 stands for the dimension and the partial ∂ symbol stands for the boundary cases, no ∂ corresponds to the bulk cases.

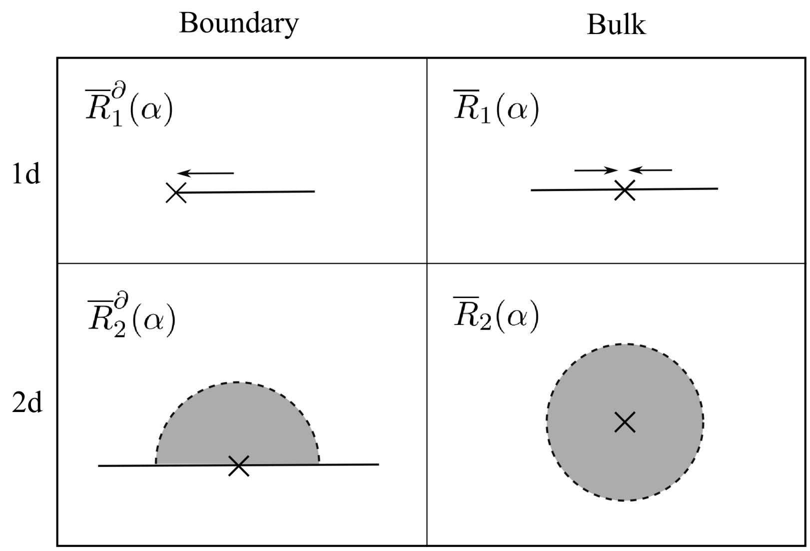

Figure 3.1: Four types of reflection coefficient

3.1.3 Small deviations for GMC

We now turn to the problem of understanding the universal behavior of the probability for a GMC to be small, both the exact formulas of Theorem 3.1.1 and the one proven on the unit circle in [70] will provide crucial insight. For this subsection we will use the following notation:

$$I_{\gamma,a,b} := \int_0^1 x^a (1-x)^b e^{\frac{\gamma}{2} X(x)} dx. \tag{3.1.32}$$

If we look at the decomposition for $I_{\gamma,a,b}$ coming from Corollary 3.1.3 it is easy to see that it is the log-normal law $L = \exp(\mathcal{N}(\gamma^2 \ln 2))$ that governs the probability to be small. One thus has,

$$\mathbb{P}(I_{\gamma,a,b} \leq \epsilon) \underset{\epsilon \to 0}{\sim} c_\epsilon e^{-\frac{(\ln \epsilon)^2}{2\gamma^2 \ln 2}}, \tag{3.1.33}$$

where c_ϵ contains lower order terms. At first glance this seems very different from the behavior found in the case of the unit circle, where the probability to be small is of the order $\exp(-\epsilon^{-\frac{4}{\gamma^2}})$. But it turns out that this difference comes from the fact that the log-correlated field on the circle is of average zero while in the case of the interval there is a non-zero global mode producing the log-normal law L. Therefore on the interval if one subtracts the average of X with respect to the correct measure (see below) one can remove the log-normal law appearing in the decomposition. The probability for the resulting GMC to be small will then match the $\exp(-\epsilon^{-\frac{4}{\gamma^2}})$ found for the model on the circle. We expect this behavior to be universal. We also deduce that the negative moments of a GMC measure will always determine its law if one can remove the global Gaussian coming from the average of the underlying field.

Let us make the above more precise. We start by writing down the decomposition of the covariance of our field in terms of the Chebyshev polynomials. For all $x, y \in [0,1]$ with $x \neq y$ we have:

$$-2\ln|x-y| = 4\ln 2 + \sum_{n=1}^{+\infty} \frac{4}{n} T_n(2x-1) T_n(2y-1). \tag{3.1.34}$$

We recall that the Chebyshev polynomial of order n is the unique polynomial verifying $T_n(\cos\theta) = T_n(\cos n\theta)$. This basis of polynomials is also orthogonal with respect to the dot product given by the integration against $\frac{1}{\sqrt{1-x^2}}dx$, i.e.

$$\int_{-1}^{1} T_n(x) T_m(x) \frac{1}{\sqrt{1-x^2}} dx = \begin{cases} 0 & \text{for} \quad n \neq m \\ \pi & \text{for} \quad n = m = 0 \\ \frac{\pi}{2} & \text{for} \quad n = m \neq 0 \end{cases} \tag{3.1.35}$$

From the above our field $X(x)$ can be constructed by the series:

$$X(x) = 2\sqrt{\ln 2}\,\alpha_0 + \sum_{n=1}^{+\infty} \frac{2\alpha_n}{\sqrt{n}} T_n(2x-1). \tag{3.1.36}$$

Here $(\alpha_n)_{n\in\mathbb{N}}$ is a sequence of i.i.d. standard Gaussian. This of course only makes sense if one integrates both sides against a test function. We now introduce:

$$\overline{X} := \int_0^1 \frac{1}{\sqrt{1-x^2}} X(x) dx = 2\sqrt{\ln 2}\,\alpha_0 \quad \text{and} \quad X_\perp(x) := X(x) - \overline{X}. \tag{3.1.37}$$

We easily check that $e^{\frac{\gamma}{2}\overline{X}} \overset{\text{law}}{=} \exp(\mathcal{N}(\gamma^2 \ln 2))$. The probability to be small for the GMC associated to $X_\perp(x)$ is now given by,

$$\mathbb{P}\Big(\int_0^1 e^{\frac{\gamma}{2} X_\perp(x)} dx \leq \epsilon\Big) \underset{\epsilon \to 0}{\sim} \tilde{c}_\epsilon e^{-\epsilon^{-\frac{4}{\gamma^2}}}, \tag{3.1.38}$$

where $\tilde{c}_\epsilon$ contains lower order terms. This result can be easily obtained from Corollary 3.1.3 by noticing that since we removed $L = \exp(\mathcal{N}(\gamma^2 \ln 2))$ the probability to be small is now governed by Y which gives (3.1.38). The argument we have just described is expected to work for any GMC in any dimension, a result of this nature will appear in [48].

There is also a direct application of these observations to determining the law of the random variable $I_{\gamma,a,b}$. This is linked to how the strategy of the proof of the present chapter differs from the one used in [70] to prove the Fyodorov-Bouchaud formula. We notice that the first differential equation (3.1.21) on $U(t)$ allows to obtain a relation between $M(\gamma, p, a, b)$ and $M(\gamma, p-1, a, b)$. Thus from this relation and knowing that $M(\gamma, 0, a, b) = 1$ one can compute recursively all the negative moments of the random variable $I_{\gamma,a,b}$. As it was emphasized in many papers (see the review [63] by Ostrovsky and references therein), the negative moments of $I_{\gamma,a,b}$ do not determine its law as the growth of the negative moments is too fast. This is this why we use the second differential equation on $\tilde{U}(t)$ to derive a second relation between $M(\gamma,p,a,b)$ and $M(\gamma,p-\frac{4}{\gamma^2},a,b)$ which gives enough information to complete the proof. By contrast in the case of the total mass of the GMC on the unit circle the negative moments do capture uniquely the probability distribution and so the proof of the Fyodorov-Bouchaud formula given in [70] only requires one differential equation (in a similar fashion one uses it to obtain a relation between the moment p and the moment $p-1$ of the total mass of the GMC).

But the negative moments of $I_{\gamma,a,b}$ do not determine its law only because of the log-normal law L in the decomposition of Corollary 3.1.3. By using Corollary 3.1.3 and by independence of $X_\perp(x)$ and $\overline{X}$ one can factor out $e^{\frac{\gamma}{2}\overline{X}} \overset{\text{law}}{=} L$ and the computation of the negative moments is now sufficient to uniquely determine the distribution. Thus the negative moments of a GMC measure always determine its law if one removes the global Gaussian coming from the average of the field with respect to an appropriate measure. From this observation the relation between $M(\gamma,p,a,b)$ and $M(\gamma,p-\frac{4}{\gamma^2},a,b)$ could be omitted in the proof of Theorem 3.1.1. Nonetheless if one only computes the negative moments it is not clear that the analytic continuation given by the Γ_γ functions does correspond to the fractional moments of a random variable, this fact has been checked by Ostrovsky in [60]. Thus in order to keep the proof of our theorem self-contained we choose to keep both shift equations.

3.1.4 Other applications

Similarly as in [70] we will write the applications of our Theorem 3.1.1 to the behaviour of the maximum of X and to random matrix theory. We refer to [70] for more detailed explanations and for additional references on these problems. Of course in the case of the unit interval the formulas are more involved than for the unit circle.

Characterizing the behaviour of the maximum of X requires to compute the law of the total mass of the derivative martingale,

$$M' := -\frac{1}{2}\int_0^1 X(x)e^{X(x)}dx, \tag{3.1.39}$$

which following [6] can be characterized by the convergence in law:

$$2M' = \lim_{\gamma\to 2}\frac{1}{2-\gamma}\int_0^1 e^{\frac{\gamma}{2}X(x)}dx. \tag{3.1.40}$$

Therefore from our Theorem 3.1.1 we can easily compute the moments of this quantity,

$$\begin{aligned}\mathbb{E}[(2M')^p] &= (2\pi)^p\frac{\Gamma_2(1-p)\Gamma_2(2-p)^2\Gamma_2(4-p)}{\Gamma_2(2)^2\Gamma_2(4-2p)}\\ &= \frac{G(4-2p)}{G(1-p)G(2-p)^2G(4-p)},\end{aligned}$$

where $G(x)$ is the so-called Barnes G function, see appendix 3.4.5 for more details. Just like in Corollary 3.1.3 an explicit description of the resulting law has been found in [62],

$$2M' \overset{law}{=} \frac{\pi}{32}LX_2X_3Y, \tag{3.1.41}$$

where L, X_2, X_3, Y are four independent random variables on $\mathbb{R}_+$ with the following laws:

$$\begin{aligned} L &= \exp(\mathcal{N}(0, 4\ln 2)) \\ X_2 &= \beta_{2,2}^{-1}(1, 1; 2, \frac{1}{2}, \frac{1}{2}) \\ X_3 &= \frac{2}{y^3} dy,\ y > 1 \\ Y &= \frac{1}{y^2} e^{-1/y},\ y > 0. \end{aligned}$$

Then for a suitable regularization X_ϵ of X the following convergence holds in law:

$$\max_{x\in[0,1]} X_\epsilon(x) - 2\ln\frac{1}{\epsilon} + \frac{3}{2}\ln\ln\frac{1}{\epsilon} \underset{\epsilon\to 0}{\to} \mathcal{G}_1 + \ln M' + C \tag{3.1.42}$$

$$= \mathcal{G}_1 + \mathcal{G}_2 + \mathcal{N}(0, 4\ln 2) + \ln X_2 + \ln X_3 + C. \tag{3.1.43}$$

All the random variables appearing above are independent, $\mathcal{G}_1$ and $\mathcal{G}_2$ are two independent Gumbel laws, and C is a non-universal real constant that depends on the regularization procedure. We have also used the fact that $\ln Y \overset{law}{=} \mathcal{G}_2$.

Lastly we briefly mention that in the case of the interval it is also possible to see the GMC measure as the limit of the characteristic polynomial of random Hermitian matrices, the connection in this case was established in [10]. The main result of [10] is that for suitable random Hermitian matrices H_N, the quantity

$$\frac{|\det(H_N - x)|^\gamma}{\mathbb{E}|\det(H_N - x)|^\gamma} dx$$

converges in law to the GMC measure on the unit interval $[0, 1]$.[6] Therefore the same applications as the ones given in [70] hold and in particular one can conjecture the following convergence in law:

$$\max_{x\in[0,1]} \ln|\det(H_N - x)| - \ln N + \frac{3}{4}\ln\ln N \underset{N\to\infty}{\to} \mathcal{G}_1 + \mathcal{G}_2 + \mathcal{N}(0, 4\ln 2) + \ln X_2 + \ln X_3 + C. \tag{3.1.44}$$

3.2 The shift equations on a and p

3.2.1 The shifts in a

The goal of this section is to prove the shift equations (3.1.10), (3.1.11) on a and b to completely determine the dependence of $M(\gamma, p, a, b)$ on these two parameters. By symmetry we will write everything only for a. We will thus prove that:

Proposition 3.2.1. *For $\gamma \in (0, 2)$ and a, b, p satisfying the bounds (3.1.5) we have,*

$$M(\gamma, p, a, b) = C(p)\frac{\Gamma_\gamma(\frac{2}{\gamma}(a+1) - (p-1)\frac{\gamma}{2})\Gamma_\gamma(\frac{2}{\gamma}(b+1) - (p-1)\frac{\gamma}{2})\Gamma_\gamma(\frac{2}{\gamma}(a+b+2) - (p-2)\frac{\gamma}{2})}{\Gamma_\gamma(\frac{2}{\gamma}(a+1) + \frac{\gamma}{2})\Gamma_\gamma(\frac{2}{\gamma}(b+1) + \frac{\gamma}{2})\Gamma_\gamma(\frac{2}{\gamma}(a+b+2) - (2p-2)\frac{\gamma}{2})}, \tag{3.2.1}$$

[6] Actually in [10] the limiting GMC measure is defined on $[-1, 1]$ but of course by a change of variable we can write everything on $[0, 1]$.

where $C(p)$ is the function that contains the remaining dependence on p (and γ). It will be computed in the section 3.2.2.

$\diamond$ *The $+\frac{\gamma^2}{4}$ shift equation*
Here we start with the first auxiliary function, for $\gamma \in (0,2)$ and a, b, p satisfying (3.1.5):

$$U(t) := U(\gamma, p; a, b; t) = \mathbb{E}[(\int_0^1 (x-t)^{\frac{\gamma^2}{4}} x^a (1-x)^b e^{\frac{\gamma}{2}X(x)} dx)^p]. \tag{3.2.2}$$

As explained in appendix 3.4.5 we can give the solutions of the hypergeometric equation for $t \in (-\infty, 0)$. We write two solutions, one corresponding to an expansion in powers of $|t|$ and one to an expansion in power of $|t|^{-1}$. Since the space of solutions is a two-dimensional vector space we will then have an explicit change of basis formula (3.4.51) linking our two solutions. We write,

$$U(t) = C_1 F(A, B, C, t) + C_2 |t|^{1-C} F(1+A-C, 1+B-C, 2-C, t) \tag{3.2.3}$$
$$= D_1 |t|^{-A} F(A, 1+A-C, 1+A-B, t^{-1}) + D_2 |t|^{-B} F(B, 1+B-C, 1+B-A, t^{-1}), \tag{3.2.4}$$

where F is the hypergeometric function. The parameters are given by:

$$A = -\frac{p\gamma^2}{4}, B = -(a+b+1) - (2-p)\frac{\gamma^2}{4}, C = -a - \frac{\gamma^2}{4}. \tag{3.2.5}$$

The idea is now to identify the constants C_1, C_2, D_1, D_2 by performing asymptotic expansions. Two of the above constants are easily obtained by evaluating $U(t)$ in $t = 0$ and by taking the limit $t \to -\infty$:

$$C_1 = M(\gamma, p, a + \frac{\gamma^2}{4}, b), \tag{3.2.6}$$
$$D_1 = M(\gamma, p, a, b). \tag{3.2.7}$$

By performing a more detailed asymptotic expansion in $t \to -\infty$ we claim that:

$$D_2 = 0. \tag{3.2.8}$$

We sketch a short proof. For $t < -2$ (arbitrary) and $x \in [0,1]$:

$$(x-t)^{\frac{\gamma^2}{4}} - |t|^{\frac{\gamma^2}{4}} \leq c|t|^{\frac{\gamma^2}{4}-1},$$

for some constant $c > 0$. By interpolating (see (3.4.21) for example), for $t < -2$,

$$\begin{aligned}|U(t) - D_1|t|^{\frac{p\gamma^2}{4}}| &\leq |p| \int_0^1 dx_1 \big((x_1-t)^{\frac{\gamma^2}{4}} - |t|^{\frac{\gamma^2}{4}}\big) x_1^a (1-x_1)^b \Big(\mathbb{E}[(\int_0^1 \frac{(x-t)^{\frac{\gamma^2}{4}} x^a (1-x)^b}{|x_1-x|^{\frac{\gamma^2}{2}}} e^{\frac{\gamma}{2}X(x)} dx)^{p-1}] \\ &\quad + \mathbb{E}[(\int_0^1 \frac{|t|^{\frac{\gamma^2}{4}} x^a (1-x)^b}{|x_1-x|^{\frac{\gamma^2}{2}}} e^{\frac{\gamma}{2}X(x)} dx)^{p-1}]\Big) \\ &\leq c'|t|^{\frac{p\gamma^2}{4}-1} M(\gamma, p, a, b) \underset{t\to-\infty}{=} O(|t|^{\frac{p\gamma^2}{4}-1}),\end{aligned}$$

where in both steps we have used the Girsanov theorem (see appendix 3.4.1) and $c' > 0$ is some constant. However, by using the bound (3.1.5) over p:

$$(-A) - (-B) = -(a + b + 1 + (2 - 2p)\frac{\gamma^2}{4}) < 1. \tag{3.2.9}$$

This implies that $D_2 = 0$. We then use the following identity coming from the theory of hypergeometric functions (3.4.51):

$$C_1 = \frac{\Gamma(1-C)\Gamma(A-B+1)}{\Gamma(A-C+1)\Gamma(1-B)} D_1. \tag{3.2.10}$$

This leads to the first shift equation (3.1.10):

$$\frac{M(\gamma, p, a + \frac{\gamma^2}{4}, b)}{M(\gamma, p, a, b)} = \frac{\Gamma(1 + a + \frac{\gamma^2}{4})\Gamma(2 + a + b - (2p-2)\frac{\gamma^2}{4})}{\Gamma(1 + a - (p-1)\frac{\gamma^2}{4})\Gamma(2 + a + b - (p-2)\frac{\gamma^2}{4})}. \tag{3.2.11}$$

⋄ *The +1 shift equation*

We now write everything with the second auxiliary function, for $\gamma \in (0, 2)$ and a, b, p satisfying (3.1.5):

$$\tilde{U} := \tilde{U}(\gamma, p; a, b; t) = \mathbb{E}[(\int_0^1 (x-t)x^a(1-x)^b e^{\frac{\gamma}{2}X(x)}dx)^p]. \tag{3.2.12}$$

Again we write the solutions of the hypergeometric equation around $t = 0_-$ and $t = -\infty$:

$$\begin{aligned}\tilde{U}(t) &= \tilde{C}_1 F(\tilde{A}, \tilde{B}, \tilde{C}, t) + \tilde{C}_2|t|^{1-\tilde{C}} F(1 + \tilde{A} - \tilde{C}, 1 + \tilde{B} - \tilde{C}, 2 - \tilde{C}, t) \\ &= \tilde{D}_1|t|^{-\tilde{A}} F(\tilde{A}, 1 + \tilde{A} - \tilde{C}, 1 + \tilde{A} - \tilde{B}, t^{-1}) + \tilde{D}_2|t|^{-\tilde{B}} F(\tilde{B}, 1 + \tilde{B} - \tilde{C}, 1 + \tilde{B} - \tilde{A}, t^{-1}).\end{aligned}$$

The parameters are given by:

$$\tilde{A} = -p, \tilde{B} = -\frac{4}{\gamma^2}(a + b + 2) + p - 1, \tilde{C} = -\frac{4}{\gamma^2}(a + 1). \tag{3.2.13}$$

Two of our constants are easily obtained,

$$\tilde{C}_1 = M(\gamma, p, a + 1, b), \tag{3.2.14}$$

$$\tilde{D}_1 = M(\gamma, p, a, b), \tag{3.2.15}$$

and we can proceed as previously to obtain:

$$\tilde{D}_2 = 0. \tag{3.2.16}$$

The relation between $\tilde{C}_1$ and $\tilde{D}_1$ (3.4.51) then leads to the shift equation (3.1.11):

$$\frac{M(\gamma, p, a + 1, b)}{M(\gamma, p, a, b)} = \frac{\Gamma(\frac{4}{\gamma^2}(1 + a) + 1)\Gamma(\frac{4}{\gamma^2}(2 + a + b) - (2p-2))}{\Gamma(\frac{4}{\gamma^2}(1 + a) - (p-1))\Gamma(\frac{4}{\gamma^2}(2 + a + b) - (p-2))}. \tag{3.2.17}$$

Therefore combining (3.2.11) and (3.2.17) proves Proposition 3.2.1.

3.2.2 The shifts in p

We now tackle the problem of determining two shift equations on p, (3.1.16) and (3.1.17) to completely determine the function $C(p)$ of Proposition 3.2.1. The idea is perform a computation at next order in the expressions of the previous subsection. This will give the desired result:

Proposition 3.2.2. *For $\gamma \in (0,2)$ and $p < \frac{4}{\gamma^2}$:*

$$C(p) = \frac{(2\pi)^p}{\Gamma(1-\frac{\gamma^2}{4})^p}(\frac{2}{\gamma})^{p\frac{\gamma^2}{4}}\frac{\Gamma_\gamma(\frac{2}{\gamma}-p\frac{\gamma}{2})}{\Gamma_\gamma(\frac{2}{\gamma})}. \tag{3.2.18}$$

◇ *The +1 shift equation*

To find the next order in $t \to 0_-$, the most natural idea is to take a such that $0 < 1-C = 1+a+\frac{\gamma^2}{4} < 1$, and then it suffices to study the equivalent of $U(t)-U(0)$ when $t \to 0_-$. For technical reasons this could only give the expression of C_2 when $\gamma < \sqrt{2}$. To obtain C_2 for all $\gamma \in (0,2)$, we will need to go one order further in the asymptotic expansion. Since we have completely determined the dependence of M on a, b by equation (3.2.1) we are free to choose these parameters as we wish and we take $0 < a < 1-\frac{\gamma^2}{4}$ and $b = 0$. In this case, we have $p < \frac{4}{\gamma^2}$, $1 < 1-C < 2$ and we perform a Taylor expansion around $t = 0_-$,

$$U(t) = U(0) + tU'(0) + t^2\int_0^1 U''(tu)(1-u)du,$$

with

$$\begin{aligned}
U''(tu) &\overset{(\star)}{=} -\frac{p\gamma^2}{4}\int_0^1 dx_1(x_1-tu)^{\frac{\gamma^2}{4}-1}x_1^a\frac{a}{x_1}\mathbb{E}[(\int_0^1 \frac{(x-tu)^{\frac{\gamma^2}{4}}x^a}{|x-x_1|^{\frac{\gamma^2}{2}}}e^{\frac{\gamma}{2}X(x)}dx)^{p-1}]\\
&= -\frac{p\gamma^2 a}{4}|tu|^{-1+a+\frac{\gamma^2}{4}}\int_0^{-\frac{1}{tu}} dy(y+1)^{\frac{\gamma^2}{4}-1}y^{a-1}\mathbb{E}[(\int_0^1 \frac{(x-tu)^{\frac{\gamma^2}{4}}x^a}{|x+tuy|^{\frac{\gamma^2}{2}}}e^{\frac{\gamma}{2}X(x)}dx)^{p-1}].
\end{aligned}$$

For $(\star)$, one may refer to (3.3.4). Next we have the following bound for $y \in [0, -\frac{1}{tu}]$, $u \in [0,1]$, and $t \in [-1,0]$:

$$\begin{aligned}
&\mathbb{E}[(\int_0^1 \frac{(x-tu)^{\frac{\gamma^2}{4}}x^a}{|x+tuy|^{\frac{\gamma^2}{2}}}e^{\frac{\gamma}{2}X(x)}dx)^{p-1}]\\
&\leq \sup_{x_1\in[0,1]}\{\mathbb{E}[(\int_0^1 \frac{x^{a+\frac{\gamma^2}{4}}}{|x-x_1|^{\frac{\gamma^2}{2}}}e^{\frac{\gamma}{2}X(x)}dx)^{p-1}] + \mathbb{E}[(\int_0^1 \frac{(x+1)^{\frac{\gamma^2}{4}}x^a}{|x-x_1|^{\frac{\gamma^2}{2}}}e^{\frac{\gamma}{2}X(x)}dx)^{p-1}]\} < \infty.
\end{aligned}$$

Then we get by dominant convergence that:

$$U''(tu) \overset{t\to 0_-}{\sim} -\frac{p\gamma^2 a}{4}|tu|^{-1+a+\frac{\gamma^2}{4}}\int_0^\infty dy(y+1)^{\frac{\gamma^2}{4}-1}y^{a-1}M(\gamma, p-1, a-\frac{\gamma^2}{4}, 0), \tag{3.2.19}$$

and again by dominant convergence:

$$\begin{aligned}
&U(t) - U(0) - tU'(0)\\
&= -\frac{p\gamma^2 a}{4}\frac{\Gamma(a+\frac{\gamma^2}{4})}{\Gamma(2+a+\frac{\gamma^2}{4})}|t|^{1+a+\frac{\gamma^2}{4}}\int_0^\infty dy(y+1)^{\frac{\gamma^2}{4}-1}y^{a-1}M(\gamma, p-1, a-\frac{\gamma^2}{4}, 0) + o(|t|^{1+a+\frac{\gamma^2}{4}}).
\end{aligned}$$

The value of the integral above is given by (3.4.59). We arrive at the expression for C_2:

$$C_2 = p\frac{\Gamma(a+1)\Gamma(-a-\frac{\gamma^2}{4}-1)}{\Gamma(-\frac{\gamma^2}{4})}M(\gamma, p-1, a-\frac{\gamma^2}{4}, 0). \tag{3.2.20}$$

The theory of hypergeometric equations (3.4.51) gives this time the relation:

$$C_2 = \frac{\Gamma(C-1)\Gamma(A-B+1)}{\Gamma(A)\Gamma(C-B)}D_1. \tag{3.2.21}$$

From this we get,

$$M(\gamma, p-1, a-\frac{\gamma^2}{4}, 0) = \frac{\Gamma(1+a-(p-1)\frac{\gamma^2}{4})\Gamma(2+a-(p-2)\frac{\gamma^2}{4})}{\Gamma(1+a)\Gamma(2+a-(2p-3)\frac{\gamma^2}{4})}M(\gamma, p-1, a, 0),$$

which gives for $0 < a < 1 - \frac{\gamma^2}{4}$ and $b = 0$ (by using the shift equation (3.2.11) on a),

$$\frac{M(\gamma, p, a, 0)}{M(\gamma, p-1, a, 0)} = \frac{\Gamma(1-\frac{p\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})}\frac{\Gamma(1+a-(p-1)\frac{\gamma^2}{4})\Gamma(1-(p-1)\frac{\gamma^2}{4})\Gamma(2+a-(p-2)\frac{\gamma^2}{4})}{\Gamma(2+a-(2p-3)\frac{\gamma^2}{4})\Gamma(2+a-(2p-2)\frac{\gamma^2}{4})}.$$

Combined with (3.2.1), this leads to a first relation on our constant $C(p)$, for $p < \frac{4}{\gamma^2}$,

$$\frac{C(p)}{C(p-1)} = \sqrt{2\pi}(\frac{\gamma}{2})^{(p-1)\frac{\gamma^2}{4}-\frac{1}{2}}\frac{\Gamma(1-p\frac{\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})}. \tag{3.2.22}$$

Reversely, (3.2.22) and (3.2.1) show that for all a, b, p satisfying the bounds (3.1.5):

$$\frac{M(\gamma, p, a, b)}{M(\gamma, p-1, a, b)} = \frac{\Gamma(1-\frac{p\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})}\frac{\Gamma(1+a-(p-1)\frac{\gamma^2}{4})\Gamma(1+b-(p-1)\frac{\gamma^2}{4})\Gamma(2+a+b-(p-2)\frac{\gamma^2}{4})}{\Gamma(2+a+b-(2p-3)\frac{\gamma^2}{4})\Gamma(2+a+b-(2p-2)\frac{\gamma^2}{4})}. \tag{3.2.23}$$

$\diamond$ *The* $+\frac{4}{\gamma^2}$ *shift equation*

Since the relation (3.2.22) is not enough to completely determine the function $C(p)$, we seek another relation on $C(p)$ that is not predicted by the Selberg integral. The techniques of this subsection are a little more involved, they lead to a relation between $C(p)$ and $C(p-\frac{4}{\gamma^2})$. Again we can pick a and b as we wish so we choose $b = 0$, a will be chosen later. The asymptotic in $t \to 0_-$ of the following quantity for $-1-\frac{\gamma^2}{4} < a < -1-\frac{\gamma^2}{8}$ is given by the lemma 3.4.9 of appendix 3.4.3:

$$\begin{aligned}&\mathbb{E}[(\int_0^1 (x-t)^{\frac{\gamma^2}{4}} x^a e^{\frac{\gamma}{2}X(x)}dx)^p] - \mathbb{E}[(\int_0^1 x^{a+\frac{\gamma^2}{4}} e^{\frac{\gamma}{2}X(x)}dx)^p] \\ =&f(\gamma, a)\frac{\Gamma(-p+1+\frac{4}{\gamma^2}(a+1))}{\Gamma(-p)}|t|^{1+a+\frac{\gamma^2}{4}}M(\gamma, p-1-\frac{4}{\gamma^2}(a+1), -2-a-\frac{\gamma^2}{4}, 0) + o(|t|^{1+a+\frac{\gamma^2}{4}}),\end{aligned}$$

where $f(\gamma, a)$ is a positive function that only depends on γ and a. Comparing with the expansion (3.2.3), we have

$$C_2 = f(\gamma, a)\frac{\Gamma(-p+1+\frac{4}{\gamma^2}(a+1))}{\Gamma(-p)}M(\gamma, p-1-\frac{4}{\gamma^2}(a+1), -2-a-\frac{\gamma^2}{4}, 0). \tag{3.2.24}$$

With the identity (3.4.51) coming from hypergeometric equations:

$$C_2 = \frac{\Gamma(C-1)\Gamma(A-B+1)}{\Gamma(A)\Gamma(C-B)}D_1 = \frac{\Gamma(-1-a-\frac{\gamma^2}{4})\Gamma(1+a-(2p-2)\frac{\gamma^2}{4})}{\Gamma(-p\frac{\gamma^2}{4})\Gamma(1-(p-1)\frac{\gamma^2}{4})}M(\gamma,p,a,0).$$

Comparing the above two expressions of C_2 yields:

$$f(\gamma,a) = \frac{M(\gamma,p,a,0)}{M(\gamma,p-1-\frac{4}{\gamma^2}(a+1),-2-a-\frac{\gamma^2}{4},0)}\frac{\Gamma(-p)\Gamma(-1-a-\frac{\gamma^2}{4})\Gamma(2+a-(2p-2)\frac{\gamma^2}{4})}{\Gamma(-p+1+\frac{4}{\gamma^2}(a+1))\Gamma(-p\frac{\gamma^2}{4})\Gamma(1-(p-1)\frac{\gamma^2}{4})}. \tag{3.2.25}$$

An interesting remark is that from (3.2.1) and analycity of the function Γ_γ, $M(\gamma,p,a,b)$ is analytic in a,b. Thus the right hand side of (3.2.25) is analytic in a. We can then deduce that the expression of the right hand side is independent from p for all appropriate a, i.e. $-1-\frac{\gamma^2}{4} < a < -1$.

In the following computations we will use the abuse of notation that $f(\gamma)$ could be a different positive function of γ every time it appears. Consider the case where $\frac{4}{k+1} < \gamma^2 < \frac{4}{k}$ for a $k \in \mathbb{N}^*$. We then take $a = -\frac{(k+1)\gamma^2}{4}$ and thus the bounds $-1-\frac{\gamma^2}{4} < a < -1$ on a are satisfied. In the previous paragraph we have shown that for $a = -\frac{(k+1)\gamma^2}{4}$:

$$\frac{M(\gamma,p,-\frac{(k+1)\gamma^2}{4},0)}{M(\gamma,p-\frac{4}{\gamma^2}+k,\frac{k\gamma^2}{4}-2,0)} = f(\gamma)\frac{\Gamma(\frac{4}{\gamma^2}-k-p)\Gamma(-p\frac{\gamma^2}{4})\Gamma(1-(p-1)\frac{\gamma^2}{4})}{\Gamma(-p)\Gamma(\frac{k\gamma^2}{4}-1)\Gamma(2-(2p+k-1)\frac{\gamma^2}{4})} \tag{3.2.26}$$

By the shift equations (3.2.11) and (3.2.17):

$$\frac{M(\gamma,p-\frac{4}{\gamma^2}+k,\frac{k\gamma^2}{4}-2,0)}{M(\gamma,p-\frac{4}{\gamma^2}+k,-\frac{(k+1)\gamma^2}{4},0)}$$
$$= f(\gamma)\prod_{j=0}^{1}\frac{\Gamma(j\frac{4}{\gamma^2}+1-p)\Gamma((1+j)\frac{4}{\gamma^2}+2-p)}{\Gamma((2+j)\frac{4}{\gamma^2}-k+2-2p)}\prod_{i=0}^{2k}\frac{\Gamma(4-(2p+3k-i-1)\frac{\gamma^2}{4})}{\Gamma(2-(p+2k-i)\frac{\gamma^2}{4})\Gamma(3-(p+2k-i-1)\frac{\gamma^2}{4})}.$$

Then by (3.2.23),

$$\frac{M(\gamma,p-\frac{4}{\gamma^2}+k,-\frac{(k+1)\gamma^2}{4},0)}{M(\gamma,p-\frac{4}{\gamma^2},-\frac{(k+1)\gamma^2}{4},0)}$$
$$= f(\gamma)\prod_{i=0}^{k-1}\frac{\Gamma(2-(p+k+i+1)\frac{\gamma^2}{4})\Gamma(2-(p+i)\frac{\gamma^2}{4})\Gamma(2-(p+1+i)\frac{\gamma^2}{4})\Gamma(3-(p+k+i)\frac{\gamma^2}{4})}{\Gamma(4-(2p+k+2i)\frac{\gamma^2}{4})\Gamma(4-(2p+k+2i+1)\frac{\gamma^2}{4})},$$

and the product of the above two equations gives:

$$\frac{M(\gamma,p-\frac{4}{\gamma^2}+k,\frac{k\gamma^2}{4}-2,0)}{M(\gamma,p-\frac{4}{\gamma^2},-\frac{(k+1)\gamma^2}{4},0)}$$
$$= f(\gamma)\frac{\Gamma(4-(2p+k-1)\frac{\gamma^2}{4})}{\Gamma(3-(p-1)\frac{\gamma^2}{4})\Gamma(3-p\frac{\gamma^2}{4})\prod_{i=0}^{k-2}(2-(p+1+i)\frac{\gamma^2}{4})}\prod_{j=0}^{1}\frac{\Gamma(j\frac{4}{\gamma^2}+1-p)\Gamma((1+j)\frac{4}{\gamma^2}+2-p)}{\Gamma((2+j)\frac{4}{\gamma^2}-k+2-2p)}.$$

Combining this relation with the previous shift equations (3.2.26):

$$\begin{aligned}
&\frac{M(\gamma,p,-\frac{(k+1)\gamma^2}{4},0)}{M(\gamma,p-\frac{4}{\gamma^2},-\frac{(k+1)\gamma^2}{4},0)} \\
&= f(\gamma)\frac{\Gamma(\frac{4}{\gamma^2}-k-p)\Gamma(-p\frac{\gamma^2}{4})\Gamma(1-(p-1)\frac{\gamma^2}{4})}{\Gamma(-p)\Gamma(\frac{k\gamma^2}{4}-1)\Gamma(2-(2p+k-1)\frac{\gamma^2}{4})}\frac{\Gamma(4-(2p+k-1)\frac{\gamma^2}{4})}{\Gamma(3-(p-1)\frac{\gamma^2}{4})\Gamma(3-p\frac{\gamma^2}{4})\prod_{i=0}^{k-2}(2-(p+1+i)\frac{\gamma^2}{4})} \\
&\times\frac{\Gamma(1-p)\Gamma(\frac{4}{\gamma^2}+1-p)\Gamma(\frac{4}{\gamma^2}+2-p)\Gamma(\frac{8}{\gamma^2}+2-p)}{\Gamma(\frac{8}{\gamma^2}-k+2-2p)\Gamma(\frac{12}{\gamma^2}-k+2-2p)} \\
&= f(\gamma)\frac{\Gamma(-p\frac{\gamma^2}{4})\Gamma(1-(p-1)\frac{\gamma^2}{4})\Gamma(4-(2p+k-1)\frac{\gamma^2}{4})\Gamma(1-p)}{\Gamma(3-p\frac{\gamma^2}{4})\Gamma(3-(p-1)\frac{\gamma^2}{4})\Gamma(2-(2p+k-1)\frac{\gamma^2}{4})\Gamma(-p)}\frac{\Gamma(\frac{8}{\gamma^2}+2-p)}{\prod_{i=0}^{k-2}(\frac{8}{\gamma^2}-(p+1+i))}\Gamma(\frac{4}{\gamma^2}+2-p) \\
&\times\frac{\Gamma(\frac{4}{\gamma^2}-k-p)\Gamma(\frac{4}{\gamma^2}+1-p)}{\Gamma(\frac{12}{\gamma^2}-k+2-2p)\Gamma(\frac{8}{\gamma^2}-k+2-2p)} \\
&= f(\gamma)\Gamma(\frac{4}{\gamma^2}-p)\frac{\Gamma(\frac{4}{\gamma^2}-k-p)\Gamma(\frac{4}{\gamma^2}+1-p)\Gamma(\frac{8}{\gamma^2}-k+1-p)}{\Gamma(\frac{12}{\gamma^2}-k+1-2p)\Gamma(\frac{8}{\gamma^2}-k+1-2p)}.
\end{aligned}$$

By (3.2.1), the same ratio of M can also be written as,

$$\frac{M(\gamma,p,-\frac{(k+1)\gamma^2}{4},0)}{M(\gamma,p-\frac{4}{\gamma^2},-\frac{(k+1)\gamma^2}{4},0)} = \frac{C(p)}{C(p-\frac{4}{\gamma^2})}f(\gamma)(\frac{\gamma}{2})^p\frac{\Gamma(\frac{4}{\gamma^2}-k-p)\Gamma(\frac{4}{\gamma^2}+1-p)\Gamma(\frac{8}{\gamma^2}-k+1-p)}{\Gamma(\frac{12}{\gamma^2}-k+1-2p)\Gamma(\frac{8}{\gamma^2}-k+1-2p)},$$

thus we obtain for $\frac{4}{k+1}<\gamma^2<\frac{4}{k}$:

$$\frac{C(p)}{C(p-\frac{4}{\gamma^2})} = f(\gamma)(\frac{\gamma}{2})^{-p}\Gamma(\frac{4}{\gamma^2}-p). \tag{3.2.27}$$

This proves the second shift equation (3.1.17) on $C(p)$. Then for every fixed γ such that $\frac{4}{\gamma^2}\notin\mathbb{Q}$ both shift equations (3.1.16) and (3.1.17) completely determine the value $C(p)$ up to a constant c_γ of γ. To see this, take another continuous function $\mathfrak{C}(p)$ that satisfies both shift equations (3.2.22) and (3.2.27). Then the ratio $\mathfrak{R}(p) := \frac{C(p)}{\mathfrak{C}(p)}$ is a 1-periodic continuous function which verifies $\mathfrak{R}(p)=\hat{c}\cdot\mathfrak{R}(p-\frac{4}{\gamma^2})$. Integrating over $[0,1]$ gives:

$$\int_0^1 dp\,\mathfrak{R}(p) = \hat{c}\cdot\int_0^1 dp\,\mathfrak{R}(p-\frac{4}{\gamma^2}). \tag{3.2.28}$$

Combining this with the 1-periodicity of $\mathfrak{R}$ yields that $\hat{c}=1$. Hence the ratio $\mathfrak{R}(p)$ is constant and $C(p)$ is determined up to a constant c_γ of γ by the two shift equations on p.

The constant c_γ is then evaluated by choosing $p=0$ and by using the known value $M(\gamma,0,a,b)=1$. Thus we arrive at the formula of Proposition 3.2.2. Finally by the continuity of $\gamma\to M(\gamma,p,a,b)$, a simple exercise, we can extend the formula to the values of γ that were left out. This completes the proof of Proposition 3.2.2.

3.3 Proof of the differential equations

We now move to the proof of Proposition 3.1.4. In order to show that $U(t)$ and $\tilde{U}(t)$ satisfy these differential equations we will need to work with a regularization procedure. A convenient way to do this is to see X as the restriction of the Gaussian field defined on the disk $\mathbb{D}+(\frac{1}{2},0)$ centered in $(\frac{1}{2},0)$ with covariance given by:

$$\mathbb{E}[X(x)X(y)] = 2\ln\frac{1}{|x-y|}. \tag{3.3.1}$$

Then for any smooth function $\theta \in \mathcal{C}^\infty([0,\infty),\mathbb{R}_+)$ with support in $[0,1]$ and satisfying $\int_0^\infty \theta = 1$, we write $\theta_\delta := \frac{1}{\delta^2}\theta(\frac{|\cdot|^2}{\delta^2})$ and define the regularized field $X_\delta := X * \theta_\delta$. Similarly we introduce:

$$\frac{1}{(x)_\delta} := \int_{\mathbb{C}}\int_{\mathbb{C}} \frac{1}{x+y_1+y_2}\theta_\delta(y_1)\theta_\delta(y_2)d^2y_1d^2y_2. \tag{3.3.2}$$

This notation will appear when we take the derivative of $\mathbb{E}[X_\delta(x)X_\delta(y)]$. We will also need some more compact notations for various quantities:

$$\begin{aligned}
G_\delta(x,y) &:= \mathbb{E}[X_\delta(x)X_\delta(y)]\\
D(x;t) &:= (x-t)^{\frac{\gamma^2}{4}}x^a(1-x)^b\\
U_{\epsilon,\delta}(t) &:= \mathbb{E}[(\int_\epsilon^{1-\epsilon} D(x;t)e^{\frac{\gamma}{2}X_\delta(x)}dx)^p]\\
V_{\epsilon,\delta}^{(1)}(x_1;t) &:= \mathbb{E}[(\int_\epsilon^{1-\epsilon} D(x;t)e^{\frac{\gamma}{2}X_\delta(x)+\frac{\gamma^2}{4}G_\delta(x,x_1)}dx)^{p-1}]\\
V_{\epsilon,\delta}^{(2)}(x_1,x_2;t) &:= \mathbb{E}[(\int_\epsilon^{1-\epsilon} D(x;t)e^{\frac{\gamma}{2}X_\delta(x)+\frac{\gamma^2}{4}(G_\delta(x,x_1)+G_\delta(x,x_2))}dx)^{p-2}]\\
E_{0,\epsilon,\delta}(t) &:= D(\epsilon;t)V_{\epsilon,\delta}^{(1)}(\epsilon;t)\\
E_{1,\epsilon,\delta}(t) &:= D(1-\epsilon;t)V_{\epsilon,\delta}^{(1)}(1-\epsilon;t).
\end{aligned}$$

The terms $V_{\epsilon,\delta}^{(1)}$ and $V_{\epsilon,\delta}^{(2)}$ will appear when we compute respectively the first and second order derivatives of $U_{\epsilon,\delta}$. The terms $E_{0,\epsilon,\delta}$ and $E_{1,\epsilon,\delta}$ are the boundary terms of the integration by parts performed below. We will also use $U_\epsilon(t)$, $V_\epsilon^{(1)}(x_1;t)$, $V_\epsilon^{(2)}(x_1,x_2;t)$, $E_{0,\epsilon}(t)$, $E_{1,\epsilon}(t)$ for the limit of the above quantities as δ goes to 0.

Proof. First we prove the equation for $U(t)$. We calculate the derivatives with the help of the Girsanov theorem of appendix 3.4.1:

$$\begin{aligned}
U'_{\epsilon,\delta}(t) =& p\int_\epsilon^{1-\epsilon} dx_1\,\partial_t D(x_1;t)V_{\epsilon,\delta}^{(1)}(x_1;t)\\
=& -p\int_\epsilon^{1-\epsilon} dx_1\partial_{x_1}((x_1-t)^{\frac{\gamma^2}{4}})x_1^a(1-x_1)^bV_{\epsilon,\delta}^{(1)}(x_1;t)\\
=& -p\Big(E_{1,\epsilon,\delta}(t)-E_{0,\epsilon,\delta}(t)-\int_\epsilon^{1-\epsilon} dx_1D(x_1;t)V_{\epsilon,\delta}^{(1)}(x_1;t)(\frac{a}{x_1}-\frac{b}{1-x_1})\\
&-\int_\epsilon^{1-\epsilon} dx_1D(x_1;t)\,\partial_{x_1}V_{\epsilon,\delta}^{(1)}(x_1;t)\Big).
\end{aligned}$$

We claim that the last term in the sum equals zero. Indeed,

$$\begin{aligned}&\int_\epsilon^{1-\epsilon} dx_1 D(x_1;t)\,\partial_{x_1} V_{\epsilon,\delta}^{(1)}(x_1;t)\\&=(p-1)\frac{\gamma^2}{2}\int_\epsilon^{1-\epsilon}\int_\epsilon^{1-\epsilon} dx_1 dx_2 \frac{D(x_1;t)D(x_2;t)}{(x_2-x_1)_\delta} e^{\frac{\gamma^2}{4}G_\delta(x_2,x_1)} V_{\epsilon,\delta}^{(2)}(x_1,x_2;t)\\&=0 \qquad \text{by symmetry.}\end{aligned}$$

Thus, by sending δ to 0,

$$U'_\epsilon(t) = -p\Big(E_{1,\epsilon}(t) - E_{0,\epsilon}(t) - \int_\epsilon^{1-\epsilon} dx_1 D(x_1;t) V_\epsilon^{(1)}(x_1;t)(\frac{a}{x_1} - \frac{b}{1-x_1})\Big). \tag{3.3.3}$$

In the same spirit, we calculate:

$$\begin{aligned}U''_{\epsilon,\delta}(t) =&\frac{p\gamma^2}{4}\Big[-\int_\epsilon^{1-\epsilon} dx_1 \partial_t\big(\frac{D(x_1;t)}{(x_1-t)}\big) V_{\epsilon,\delta}^{(1)}(x_1;t)\\&+\frac{(p-1)\gamma^2}{4}\int_\epsilon^{1-\epsilon} dx_1 \int_\epsilon^{1-\epsilon} dx_2 \frac{D(x_1;t)D(x_2;t)}{(x_1-t)(x_2-t)} e^{\frac{\gamma^2}{4}G_\delta(x_2,x_1)} V_{\epsilon,\delta}^{(2)}(x_1,x_2;t)\Big].\end{aligned}$$

An integration by parts gives:

$$\begin{aligned}&-\int_\epsilon^{1-\epsilon} dx_1 \partial_t\big(\frac{D(x_1;t)}{(x_1-t)}\big) V_{\epsilon,\delta}^{(1)}(x_1;t)\\&=\int_\epsilon^{1-\epsilon} dx_1 \partial_{x_1}\big(\frac{|x_1-t|^{\frac{\gamma^2}{4}}}{x_1-t}\big) x_1^a (1-x_1)^b V_{\epsilon,\delta}^{(1)}(x_1;t)\\&=\frac{1}{1-t-\epsilon}E_{1,\epsilon,\delta}(t) + \frac{1}{t-\epsilon}E_{0,\epsilon,\delta}(t) - \int_\epsilon^{1-\epsilon} dx_1 D(x_1;t) V_{\epsilon,\delta}^{(1)}(x_1;t)\frac{1}{x_1-t}(\frac{a}{x_1} - \frac{b}{1-x_1})\\&\quad - \frac{(p-1)\gamma^2}{2}\int_\epsilon^{1-\epsilon} dx_1 \int_\epsilon^{1-\epsilon} dx_2 \frac{D(x_1;t)D(x_2;t)}{(x_1-t)(x_2-x_1)_\delta} e^{\frac{\gamma^2}{4}G_\delta(x_2,x_1)} V_{\epsilon,\delta}^{(2)}(x_1,x_2;t).\end{aligned}$$

By symmetry of the expression under the exchange of x_1 and x_2,

$$\begin{aligned}&\frac{(p-1)\gamma^2}{2}\int_\epsilon^{1-\epsilon} dx_1 \int_\epsilon^{1-\epsilon} dx_2 \frac{D(x_1;t)D(x_2;t)}{(x_1-t)(x_2-x_1)_\delta} e^{\frac{\gamma^2}{4}G_\delta(x_2,x_1)} V_{\epsilon,\delta}^{(2)}(x_1,x_2;t)\\&=\frac{(p-1)\gamma^2}{4}\int_\epsilon^{1-\epsilon} dx_1 \int_\epsilon^{1-\epsilon} dx_2 D(x_1;t)D(x_2;t) e^{\frac{\gamma^2}{4}G_\delta(x_2,x_1)}\\&\quad\times\big(\frac{1}{(x_1-t)(x_2-x_1)_\delta} + \frac{1}{(x_2-t)(x_1-x_2)_\delta}\big) V_{\epsilon,\delta}^{(2)}(x_1,x_2;t)\\&=\frac{(p-1)\gamma^2}{4}\int_\epsilon^{1-\epsilon} dx_1 \int_\epsilon^{1-\epsilon} dx_2 \frac{D(x_1;t)D(x_2;t)}{(x_1-t)(x_2-t)}\frac{x_2-x_1}{(x_2-x_1)_\delta} e^{\frac{\gamma^2}{4}G_\delta(x_2,x_1)} V_{\epsilon,\delta}^{(2)}(x_1,x_2;t).\end{aligned}$$

Since $\frac{x_2-x_1}{(x_2-x_1)_\delta} \leq c$ for some constant $c>0$ independent of δ, by tending δ to 0,

$$U''_\epsilon(t) = \frac{p\gamma^2}{4}\Big(\frac{1}{1-t-\epsilon}E_{1,\epsilon}(t) + \frac{1}{t-\epsilon}E_{0,\epsilon}(t) - \int_\epsilon^{1-\epsilon} dx_1 D(x_1;t) V_\epsilon^{(1)}(x_1;t)\frac{1}{x_1-t}(\frac{a}{x_1} - \frac{b}{1-x_1})\Big). \tag{3.3.4}$$

A further calculation shows that,

$$\begin{aligned}&\int_{\epsilon}^{1-\epsilon} dx_1 D(x_1;t)V_{\epsilon}^{(1)}(x_1;t)\frac{1}{x_1-t}(\frac{a}{x_1}-\frac{b}{1-x_1})\\&=\int_{\epsilon}^{1-\epsilon} dx_1 D(x_1;t)V_{\epsilon}^{(1)}(x_1;t)\Big(\frac{a}{t}(\frac{1}{x_1-t}-\frac{1}{x_1})-\frac{b}{1-t}(\frac{1}{x_1-t}+\frac{1}{1-x_1})\Big)\\&=-\int_{\epsilon}^{1-\epsilon} dx_1 D(x_1;t)V_{\epsilon}^{(1)}(x_1;t)(\frac{a}{tx_1}+\frac{b}{(1-t)(1-x_1)})-\frac{4}{p\gamma^2}(\frac{a}{t}-\frac{b}{1-t})U_{\epsilon}'(t),\end{aligned}$$

and as a consequence,

$$\begin{aligned}U_{\epsilon}''(t)=&\frac{p\gamma^2}{4}\Big(\frac{1}{1-t-\epsilon}E_{1,\epsilon}(t-\epsilon)+\frac{1}{t}E_{0,\epsilon}(t)+\int_{\epsilon}^{1-\epsilon} dx_1 D(x_1;t)V_{\epsilon}^{(1)}(x_1;t)(\frac{a}{tx_1}+\frac{b}{(1-t)(1-x_1)})\Big)\\&+(\frac{a}{t}-\frac{b}{1-t})U_{\epsilon}'(t).\end{aligned} \tag{3.3.5}$$

We can also write $U_{\epsilon,\delta}(t)$ in a similar form, by doing an integration by part:

$$\begin{aligned}&(1-t-\epsilon)E_{1,\epsilon,\delta}(t)+(t-\epsilon)E_{0,\epsilon,\delta}(t)-\int_{\epsilon}^{1-\epsilon} dx_1 D(x_1;t)V_{\epsilon,\delta}^{(1)}(x_1;t)(x_1-t)(\frac{a}{x_1}-\frac{b}{1-x_1})\\&=(1+\frac{\gamma^2}{4})\int_{\epsilon}^{1-\epsilon} dx_1 D(x_1;t)V_{\epsilon,\delta}^{(1)}(x_1;t)\\&\quad+(p-1)\frac{\gamma^2}{2}\int_{\epsilon}^{1-\epsilon}\int_{\epsilon}^{1-\epsilon} dx_1dx_2 D(x_1;t)D(x_2;t)e^{\frac{\gamma^2}{4}G_{\delta}(x_2,x_1)}\frac{x_1-t}{(x_2-x_1)_{\delta}}V_{\epsilon,\delta}^{(2)}(x_1,x_2;t)\\&=(1+\frac{\gamma^2}{4})\int_{\epsilon}^{1-\epsilon} dx_1 D(x_1;t)V_{\epsilon,\delta}^{(1)}(x_1;t)\\&\quad-(p-1)\frac{\gamma^2}{4}\int_{\epsilon}^{1-\epsilon} dx_1\int_{\epsilon}^{1-\epsilon} dx_2 D(x_1;t)D(x_2;t)e^{\frac{\gamma^2}{4}G_{\delta}(x_2,x_1)}\frac{x_2-x_1}{(x_2-x_1)_{\delta}}V_{\epsilon,\delta}^{(2)}(x_1,x_2;t).\end{aligned}$$

By sending δ to 0 and by applying the Girsanov theorem of appendix 3.4.1, we obtain:

$$\begin{aligned}-(B+a+b)U_{\epsilon}(t)=&(1-t-\epsilon)E_{1,\epsilon}(t)+(t-\epsilon)E_{0,\epsilon}(t)\\&-\int_{\epsilon}^{1-\epsilon} dx_1 D(x_1;t)V_{\epsilon}^{(1)}(x_1;t)(x_1-t)(\frac{a}{x_1}-\frac{b}{1-x_1}).\end{aligned}$$

We also note that,

$$\begin{aligned}&\int_{\epsilon}^{1-\epsilon} dx_1 D(x_1;t)V_{\epsilon}^{(1)}(x_1;t)(x_1-t)(\frac{a}{x_1}-\frac{b}{1-x_1})\\&\quad=(a+b)U_{\epsilon,\delta}(t)-\int_{\epsilon}^{1-\epsilon} dx_1 D(x_1;t)V_{\epsilon}^{(1)}(x_1;t)(\frac{at}{x_1}+\frac{b(1-t)}{1-x_1}),\end{aligned}$$

and hence,

$$-BU_{\epsilon}(t)=(1-t-\epsilon)E_{1,\epsilon}(t)+(t-\epsilon)E_{0,\epsilon}(t)+\int_{\epsilon}^{1-\epsilon} dx_1 D(x_1;t)V_{\epsilon}^{(1)}(x_1;t)(\frac{at}{x_1}+\frac{b(1-t)}{1-x_1}). \tag{3.3.6}$$

Combining this with the expressions for U'_ϵ and U''_ϵ, equations (3.3.3) and (3.3.5):

$$U'_\epsilon(t) = -p\Big(E_{1,\epsilon}(t) - E_{0,\epsilon}(t) - \int_\epsilon^{1-\epsilon} dx_1 D(x_1;t)V_\epsilon^{(1)}(x_1;t)(\frac{a}{x_1} - \frac{b}{1-x_1})\Big),$$
$$U''_\epsilon(t) = \frac{p\gamma^2}{4}\Big(\frac{1}{1-t-\epsilon}E_{1,\epsilon}(t) + \frac{1}{t-\epsilon}E_{0,\epsilon}(t) + \int_\epsilon^{1-\epsilon} dx_1 D(x_1;t)V_\epsilon^{(1)}(x_1;t)(\frac{a}{tx_1} + \frac{b}{(1-t)(1-x_1)})\Big)$$
$$+ (\frac{a}{t} - \frac{b}{1-t})U'_{\epsilon,\delta}(t),$$

we finally arrive at:

$$t(1-t)U''_\epsilon(t) + (C-(A+B+1)t)U'_\epsilon(t) - ABU_\epsilon(t) = \epsilon(1-\epsilon)\frac{p\gamma^2}{4}(\frac{1}{1-t-\epsilon}E_{1,\epsilon}(t) + \frac{1}{t-\epsilon}E_{0,\epsilon}(t)). \tag{3.3.7}$$

From this expression we see that the last thing we need to check is that as ϵ goes to zero the right hand side of the above expression converges to 0 in a suitable sense. Indeed we will prove that, for t in a fixed compact set $K \subseteq (-\infty, 0)$, $\epsilon E_{1,\epsilon}(t)$ and $\epsilon E_{0,\epsilon}(t)$ converge uniformly to 0 for a well chosen sequence of ϵ. Let us consider $\epsilon E_{0,\epsilon}(t)$ as $\epsilon E_{1,\epsilon}(t)$ can be treated in a similar fashion:

$$\epsilon E_{0,\epsilon}(t) = (\epsilon - t)^{\frac{\gamma^2}{4}}\epsilon^{a+1}(1-\epsilon)^b\mathbb{E}[(\int_\epsilon^{1-\epsilon}\frac{(x-t)^{\frac{\gamma^2}{4}}x^a(1-x)^b}{|x-\epsilon|^{\frac{\gamma^2}{2}}}e^{\frac{\gamma}{2}X(x)}dx)^{p-1}].$$

In the following we will discuss three disjoint cases based on the value of a, they are $a > -1 + \frac{\gamma^2}{4}$, $-1 < a \leq -1 + \frac{\gamma^2}{4}$, and $-1 - \frac{\gamma^2}{4} < a \leq -1$.

i) $a > -1 + \frac{\gamma^2}{4}$

This is the simplest case as we have for ϵ sufficiently small,

$$\epsilon E_{0,\epsilon}(t) \leq C_0\epsilon^{a+1}(1-\epsilon)^b\mathbb{E}[(\int_0^1\frac{x^a(1-x)^b}{|x-\epsilon|^{\frac{\gamma^2}{2}}}e^{\frac{\gamma}{2}X(x)}dx)^{p-1}] \overset{\epsilon\to 0}{\sim} C_0\epsilon^{a+1}M(\gamma, p, a - \frac{\gamma^2}{2}, b),$$

which converges to 0 as $\epsilon \to 0$ uniformly over $t \in K$.

ii) $-1 < a \leq -1 + \frac{\gamma^2}{4}$.

In this case we have $p - 1 < 1$ and $\epsilon^{a+1} \underset{\epsilon\to 0}{\longrightarrow} 0$. If $p-1 \leq 0$, $\mathbb{E}[(\int_\epsilon^{1-\epsilon}\frac{(x-t)^{\frac{\gamma^2}{4}}x^a(1-x)^b}{|x-\epsilon|^{\frac{\gamma^2}{2}}}e^{\frac{\gamma}{2}X(x)}dx)^{p-1}]$ is uniformly bounded thus it is immediate to obtain the convergence to 0. Hence it suffices to consider the case $0 < p-1 < 1$. We choose $\epsilon_N = \frac{1}{2^N}$. Using the sub-additivity of the function $x \mapsto x^{p-1}$, we have for some $C_0, C' > 0$ independent of K:

$$\epsilon_N E_{0,\epsilon_N}(t) \leq C_0\epsilon_N^{a+1}\mathbb{E}[(\int_{\epsilon_N}^{\frac{1}{2}}\frac{x^a}{|x-\epsilon_N|^{\frac{\gamma^2}{2}}}e^{\frac{\gamma}{2}X(x)}dx)^{p-1}] + C'\epsilon_N^{a+1}$$
$$\leq C_0\epsilon_N^{a+1}\sum_{n=1}^{N-1}\mathbb{E}[(\int_{\epsilon_{n+1}}^{\epsilon_n}\frac{x^a}{|x-\epsilon_{n+1}|^{\frac{\gamma^2}{2}}}e^{\frac{\gamma}{2}X(x)}dx)^{p-1}] + C'\epsilon_N^{a+1}.$$

Then by a scaling property of GMC,

$$\begin{aligned}
&\mathbb{E}[(\int_{\epsilon_{n+1}}^{\epsilon_n} \frac{x^a}{|x-\epsilon_{n+1}|^{\frac{\gamma^2}{2}}} e^{\frac{\gamma}{2}X(x)}dx)^{p-1}] \\
=&2^{\frac{\gamma^2}{4}(p-1)(p-2)-(a-\frac{\gamma^2}{2}+1)(p-1)}\mathbb{E}[(\int_{\epsilon_n}^{\epsilon_{n-1}} \frac{u^a}{|u-\epsilon_n|^{\frac{\gamma^2}{2}}} e^{\frac{\gamma}{2}X(u)}du)^{p-1}] \\
=&2^{\frac{\gamma^2}{4}p^2-(\frac{\gamma^2}{4}+a+1)p+a+1}\mathbb{E}[(\int_{\epsilon_n}^{\epsilon_{n-1}} \frac{u^a}{|u-\epsilon_n|^{\frac{\gamma^2}{2}}} e^{\frac{\gamma}{2}X(u)}du)^{p-1}].
\end{aligned}$$

We can deduce that:

$$\begin{aligned}
\epsilon_N E_{0,\epsilon_N}(t) \leq& C_0 2^{-N(a+1)} 2^{(N-1)(\frac{\gamma^2}{4}p^2-(\frac{\gamma^2}{4}+a+1)p+a+1)}\mathbb{E}[(\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{x^a}{|x-\frac{1}{4}|^{\frac{\gamma^2}{2}}} e^{\frac{\gamma}{2}X(x)}dx)^{p-1}] + C'\epsilon_N^{a+1} \\
\leq& C 2^{N(\frac{\gamma^2}{4}p-\frac{\gamma^2}{4}-a-1)p} + C'\epsilon_N^{a+1} \overset{N\to\infty}{\longrightarrow} 0.
\end{aligned}$$

for some $C, C' > 0$. The convergence holds since $p > 0$ and $\frac{\gamma^2}{4}p - \frac{\gamma^2}{4} - a - 1 < 0$ (this inequality comes from (3.1.5)), and it holds uniformly over t in K.

iii) $-1 - \frac{\gamma^2}{4} < a \leq -1$

In this case $p - 1 < 0$ so we are always dealing with negative moments. This implies that for t in K, we can bound $\epsilon E_{0,\epsilon}(t)$ by,

$$\epsilon E_{0,\epsilon}(t) \leq C_0 \epsilon^{a+1}\mathbb{E}[(\int_{\epsilon}^{\frac{1}{2}} x^{a-\frac{\gamma^2}{2}} e^{\frac{\gamma}{2}X(x)}dx)^{p-1}],$$

simply by restricting the integral over $[\epsilon, 1-\epsilon]$ to $[\epsilon, 1/2]$. An estimation of the resulting GMC moment is given by lemma 3.4.4 in appendix 3.4.2: for ϵ sufficiently small,

$$\mathbb{E}[(\int_{\epsilon}^{\frac{1}{2}} x^{a-\frac{\gamma^2}{2}} e^{\frac{\gamma}{2}X(x)}dx)^{p-1}] \leq C\sqrt{\ln(1/\epsilon)}\,\epsilon^{\alpha^2},$$

with $\alpha = \frac{\gamma}{4} - \frac{1}{\gamma}(a+1)$. This suffices to show the convergence to 0 of $\epsilon E_{0,\epsilon}(t)$. Indeed, a basic inequality shows that $\alpha^2 \geq -(a+1)$ with equality when $-(a+1) = \frac{\gamma^2}{4}$. Since the condition cannot be satisfied, we have the strict inequality $\alpha^2 > -(a+1)$. Hence,

$$\epsilon E_{0,\epsilon}(t) \leq C'\epsilon^{a+1+\alpha^2}\sqrt{\ln(1/\epsilon)} \overset{\epsilon\to 0}{\longrightarrow} 0,$$

where the convergence is again uniform over t in K. Combining the cases (i), (ii) and (iii), we have proven the differential equation 3.1.21 in the weak sense. Since it is a hypoelliptic equation (the dominant operator is a Laplacian) with analytic coefficients, $U(t)$ is analytic and the equation holds in the strong sense.

Let us now briefly mention the case of $\tilde{U}(t)$. In a similar manner, we calculate,

$$-\tilde{B}\tilde{U}(t) = \frac{4}{\gamma^2}\Big((1-t-\epsilon)\tilde{E}_{1,\epsilon}(t) + (t-\epsilon)\tilde{E}_{0,\epsilon}(t) + \int_\epsilon^{1-\epsilon} dx_1 \tilde{D}(x_1;t)\tilde{V}_\epsilon^{(1)}(x_1;t)(\frac{at}{x_1} + \frac{b(1-t)}{1-x_1})\Big),$$
$$\tilde{U}'_\epsilon(t) = -p\Big(\tilde{E}_{1,\epsilon}(t) - \tilde{E}_{0,\epsilon}(t) - \int_\epsilon^{1-\epsilon} dx_1 \tilde{D}(x_1;t)\tilde{V}_\epsilon^{(1)}(x_1;t)(\frac{a}{x_1} - \frac{b}{1-x_1})\Big),$$
$$\tilde{U}''_\epsilon(t) = \frac{4p}{\gamma^2}\Big(\frac{1}{1-t-\epsilon}\tilde{E}_{1,\epsilon}(t) + \frac{1}{t-\epsilon}\tilde{E}_{0,\epsilon}(t) + \int_\epsilon^{1-\epsilon} dx_1 \tilde{D}(x_1;t)\tilde{V}_\epsilon^{(1)}(x_1;t)(\frac{a}{tx_1} + \frac{b}{(1-t)(1-x_1)})\Big)$$
$$+ \frac{4}{\gamma^2}(\frac{a}{t} - \frac{b}{1-t})\tilde{U}'_\epsilon(t),$$

where $\tilde{D}(x;t) := (x-t)\,x^a(1-x)^b$ and $\tilde{V}_\epsilon^{(1)}(x_1;t)$, $\tilde{E}_{0,\epsilon}(t)$, $\tilde{E}_{1,\epsilon}(t)$ are defined as functions of $\tilde{D}(x;t)$, the same as their definitions without the tilde. We verify easily that,

$$t(1-t)\tilde{U}''_\epsilon(t) + (\tilde{C} - (\tilde{A}+\tilde{B}+1)t)\tilde{U}'_\epsilon(t) - \tilde{A}\tilde{B}\tilde{U}_\epsilon(t) = \epsilon(1-\epsilon)\frac{p\gamma^2}{4}(\frac{1}{1-t-\epsilon}\tilde{E}_{1,\epsilon}(t) + \frac{1}{t-\epsilon}\tilde{E}_{0,\epsilon}(t)), \tag{3.3.8}$$

and the right hand side of the above expression converges again to zero uniformly for t in any compact set of $(-\infty, 0)$, which finishes the proof. □

3.4 Appendix

3.4.1 Reminder on some useful theorems

We recall some theorems in probability that we will use without further justification. In the following, D is a compact subset of $\mathbb{R}^d$.

Theorem 3.4.1 (Girsanov theorem). *Let $(Z(x))_{x\in D}$ be a continuous centered Gaussian process and Z a Gaussian variable which belongs to the L^2 closure of the vector space spanned by $(Z(x))_{x\in D}$. Let F be some continuous bounded function on $\mathcal{C}(D,\mathbb{R})$. Then we have the following identity:*

$$\mathbb{E}[e^{Z-\frac{\mathbb{E}[Z^2]}{2}}F((Z(x))_{x\in D})] = \mathbb{E}[F((Z(x)+\mathbb{E}[Z(x)Z])_{x\in D})]. \tag{3.4.1}$$

When applied to our case, although the log-correlated field X is not a continuous Gaussian process, we can still make the arguments rigorous by using a regularization procedure. Let us illustrate the idea by a simple example:

$$\mathbb{E}[(\int_0^1 x^a(1-x)^b e^{\frac{\gamma}{2}X(x)}dx)^p]$$
$$= \lim_{\epsilon\to 0}\lim_{\delta\to 0}\int_\epsilon^{1-\epsilon} dx_1\, x_1^a(1-x_1)^b \mathbb{E}[e^{\frac{\gamma}{2}X_\delta(x_1)-\frac{\gamma^2}{8}\mathbb{E}[X_\delta(x_1)^2]}(\int_\epsilon^{1-\epsilon} x^a(1-x)^b e^{\frac{\gamma}{2}X_\delta(x)-\frac{\gamma^2}{8}\mathbb{E}[X_\delta(x)^2]}dx)^{p-1}]$$
$$\overset{(3.4.1)}{=} \lim_{\epsilon\to 0}\lim_{\delta\to 0}\int_\epsilon^{1-\epsilon} dx_1\, x_1^a(1-x_1)^b \mathbb{E}[(\int_\epsilon^{1-\epsilon} x^a(1-x)^b e^{\frac{\gamma}{2}X_\delta(x)+\frac{\gamma^2}{4}\mathbb{E}[X_\delta(x)X_\delta(x_1)]-\frac{\gamma^2}{8}\mathbb{E}[X_\delta(x)^2]}dx)^{p-1}]$$
$$= \int_0^1 dx_1\, x_1^a(1-x_1)^b \mathbb{E}[(\int_0^1 \frac{x^a(1-x)^b}{|x_1-x|^{\frac{\gamma^2}{2}}} e^{\frac{\gamma}{2}X(x)}dx)^{p-1}].$$

The next theorem is a comparison result due to Kahane [42]:

Theorem 3.4.2 (Convexity inequality)**.** *Let $(Z_1(x))_{x\in D}$, $(Z_2(x))_{x\in D}$ be two continuous centered Gaussian processes such that for all $x, y \in D$:*

$$\mathbb{E}[Z_1(x)Z_1(y)] \leq \mathbb{E}[Z_2(x)Z_2(y)].$$

Then for all convex function (resp. concave) F with at most polynomial growth at infinity, and σ a positive finite measure over D,

$$\mathbb{E}[F(\int_D e^{Z_1(x)-\frac{1}{2}\mathbb{E}[Z_1(x)^2]}\sigma(dx))] \leq (\text{ resp. } \geq) \, \mathbb{E}[F(\int_D e^{Z_2(x)-\frac{1}{2}\mathbb{E}[Z_2(x)^2]}\sigma(dx))]. \tag{3.4.2}$$

To apply this theorem to log-correlated fields, one needs again to use a regularization procedure. Finally, we provide the Williams decomposition theorem, see for instance [85]:

Theorem 3.4.3. *Let $(B_s - vs)_{s\geq 0}$ be a Brownian motion with negative drift, i.e. $v > 0$ and let $M = \sup_{s\geq 0}(B_s - vs)$. Then conditionally on M the law of the path $(B_s - vs)_{s\geq 0}$ is given by the joining of two independent paths:*
1) A Brownian motion $(B^1_s + vs)_{0\leq s\leq \tau_M}$ with positive drift v run until its hitting time τ_M of M.
2) $(M + B^2_t - vt)_{t\geq 0}$ where $(B^2_t - vt)_{t\geq 0}$ is a Brownian motion with negative drift conditioned to stay negative.

Moreover, one has the following time reversal property for all $C > 0$ (where τ_C denotes the hitting time of C),

$$(B^1_{\tau_C - s} + v(\tau_C - s) - C)_{0\leq s\leq \tau_C} \overset{law}{=} (\tilde{B}_s - vs)_{0\leq s\leq L_{-C}}, \tag{3.4.3}$$

where $(\tilde{B}_s - vs)_{s\geq 0}$ is a Brownian motion with drift $-v$ conditioned to stay negative and L_{-C} is the last time $(\tilde{B}_s - vs)_{s\geq 0}$ hits $-C$.

3.4.2 An estimate on GMC

Just like in section 3.1.2 for $s \geq 0$ we write $X(e^{-s/2}) = B_s + Y(e^{-s/2})$ where B_s is a standard Brownian motion and Y is an independent centered Gaussian field on $\mathbb{C}$ with covariance:

$$\mathbb{E}[Y(x)Y(y)] = 2\ln\frac{|x|\vee|y|}{|x-y|}. \tag{3.4.4}$$

Denote the GMC measure associated to $Y(e^{-s/2})$ by $\mu_Y(ds) := e^{\frac{\gamma}{2}Y(e^{-s/2})}ds$. The goal of this subsection is to prove the following lemma:

Lemma 3.4.4. *For $q > 0$, $a < -1-\frac{\gamma^2}{4}$, and a fixed constant $A > 0$, there exists $\epsilon_1 < A$ sufficiently small such that for all $\epsilon \leq \epsilon_1$,*

$$\mathbb{E}[(\int_\epsilon^A x^a e^{\frac{\gamma}{2}X(x)}dx)^{-q}] \leq C\sqrt{\ln(1/\epsilon)}\,\epsilon^{(\frac{\gamma}{4}+\frac{1}{\gamma}(a+1))^2}, \tag{3.4.5}$$

where $C > 0$ is a constant independent of ϵ.

By using the decomposition described above, we can transform this lemma into another equivalent form,

$$\begin{aligned}\mathbb{E}[(\int_\epsilon^A x^a e^{\frac{\gamma}{2}X(x)}dx)^{-q}] =& 2^q\mathbb{E}[(\int_{-2\ln A}^{-2\ln\epsilon} e^{\frac{\gamma}{2}(B_s - s(\frac{\gamma}{4}+\frac{1}{\gamma}(a+1)))}\mu_Y(ds))^{-q}] \\ =& 2^q\mathbb{E}[(\int_{-2\ln A}^{-2\ln\epsilon} e^{\frac{\gamma}{2}(B_s+\alpha s)}\mu_Y(ds))^{-q}],\end{aligned}$$

where again $(B_s)_{s\geq 0}$ is a standard Brownian motion independent from Y, and $\alpha = -\frac{\gamma}{4} - \frac{1}{\gamma}(a+1)$. Now we can focus on the following lemma:

Lemma 3.4.5. *For $q > 0$, $\alpha > 0$, and a fixed constant r_0, there exists $r_1 > r_0$ sufficiently large such that for all $r \geq r_1$,*

$$\mathbb{E}[(\int_{r_0}^r e^{\frac{\gamma}{2}(B_s+\alpha s)}\mu_Y(ds))^{-q}] \leq C\sqrt{r}e^{-\frac{\alpha^2 r}{2}}, \tag{3.4.6}$$

where $C > 0$ is a constant independent of r.

A similar result for 2d GMC has been proved in [46] (proposition 5.1). A slight difference is that in [46] the power q depends on a.

We start by proving three intermediate results. We denote $y_s = B_s + \alpha s$, and we introduce for $\beta \geq 1$ the stopping time $T_\beta = \inf\{s \geq 0, y_s = \beta - 1\}$. Recall the density of T_β for $\beta > 1, u > 0$:

$$\mathbb{P}(T_\beta \in (u, u+du)) = \frac{\beta - 1}{\sqrt{2\pi}u^{3/2}}e^{-\frac{(\beta-1-\alpha u)^2}{2u}}du. \tag{3.4.7}$$

Lemma 3.4.6. *For $\alpha, A > 0$, we have:*

$$\mathbb{P}(\sup_{s\leq t} y_s \leq A) \leq e^{\alpha A - \frac{\alpha^2 t}{2}}. \tag{3.4.8}$$

Proof. We know the density of $\sup_{s\leq t} y_s$:

$$\mathbb{P}(\sup_{s\leq t} y_s \leq A) = \mathbb{P}(T_{A+1} \geq t) = \frac{A}{\sqrt{2\pi}}\int_t^\infty \frac{e^{-\frac{(A-\alpha s)^2}{2s}}}{s^{3/2}}ds \leq \frac{Ae^{\alpha A - \frac{\alpha^2 t}{2}}}{\sqrt{2\pi}}\int_0^\infty \frac{e^{-\frac{A^2}{2s}}}{s^{3/2}}ds = e^{\alpha A - \frac{\alpha^2 t}{2}}.$$

□

Lemma 3.4.7. *We set for $t > 0$:*

$$I(t) = \int_t^{t+1} e^{-\frac{\gamma}{2}(y_s - y_t)}\mu_Y(ds). \tag{3.4.9}$$

For $q > 0$, we have the following inequality,

$$\mathbb{E}[I(t)^{-q}|y_{t+1} - y_t] \leq C_1(e^{-\frac{\gamma}{2}q(y_{t+1}-y_t)} + 1) \quad a.s., \tag{3.4.10}$$

where C_1 depends on q, γ.

Proof. Conditioning on $y_{t+1} - y_t = y$, $(B_s - B_t)_{t \le s \le t+1}$ has the law of a Brownian bridge between 0 and $y - \alpha$. Hence it has the law of $(B'_s - sB'_1 + s(y-\alpha))_{0 \le s \le 1}$, where B' is an independent Brownian motion. We have:

$$\mathbb{E}[I(t)^{-q} | y_{t+1} - y_t = y] = \mathbb{E}[(\int_0^1 e^{\frac{\gamma}{2}(B'_s - sB'_1 + sy)} \mu_Y(ds))^{-q}].$$

Notice that $e^{\frac{\gamma}{2}sy} \ge e^{\frac{\gamma}{2}y} \wedge 1$, and a classic result on the moments of Gaussian multiplicative chaos shows that

$$\mathbb{E}[(\mu_Y(ds))^{-q}] < \infty,$$

thus:

$$\mathbb{E}[(\int_0^1 e^{\frac{\gamma}{2}(B'_s - sB'_1)} \mu_Y(ds))^{-q}] \le \mathbb{E}[e^{-\frac{q\gamma}{2}\inf_{0 \le s \le 1}(B'_s - sB'_1)}]\mathbb{E}[(\mu_Y(ds))^{-q}] = C_1 < \infty.$$

We can now derive that:

$$\mathbb{E}[I(t)^{-q} | y_{t+1} - y_t = y] \le C_1(e^{-\frac{\gamma}{2}qy} \vee 1) \le C_1(e^{-\frac{\gamma}{2}qy} + 1) \quad a.s.$$

□

Lemma 3.4.8. *Define for $q > 0$, $\beta \ge 1$, $\alpha > 0$ and $r > 1$,*

$$J_{r,\beta} := \mathbb{E}[\frac{\mathbf{1}_{\{\sup_{s\in[0,r]} y_s \in [\beta-1,\beta]\}}}{(\int_0^r e^{\frac{\gamma}{2}y_s} \mu_Y(ds))^q}], \tag{3.4.11}$$

then there exists a constant $C_2 > 0$ such that

$$J_{r,\beta} \le C_2 e^{-\frac{\alpha^2 r}{2}} e^{(\alpha - \frac{q\gamma}{2})\beta - \frac{(\beta-1)^2}{2r}} (1 + \frac{\beta - 1}{(r-1)^{3/2}}), \tag{3.4.12}$$

where C_2 is independent of β, r.

Proof.

$$J_{r,\beta} \le e^{-\frac{q\gamma(\beta-1)}{2}} \mathbb{E}[\mathbf{1}_{\{T_\beta \le r-1\}} \frac{\mathbf{1}_{\{\sup_{s\in[0,r]} y_s \in [\beta-1,\beta]\}}}{I(T_\beta)^q}] + \mathbb{E}[\mathbf{1}_{\{T_\beta > r-1\}} \frac{\mathbf{1}_{\{\sup_{s\in[0,r]} y_s \in [\beta-1,\beta]\}}}{e^{\frac{q\gamma y_{r-1}}{2}} I(r-1)^q}] =: A + B.$$

We first bound A. By using the strong Markov property of $(y_s)_{s\ge 0}$ with respect to $\mathcal{F}_{T_\beta+1}$:

$$\begin{aligned} A \le & e^{-\frac{q\gamma(\beta-1)}{2}} \mathbb{E}[\mathbf{1}_{\{T_\beta+1 \le r\}} I(T_\beta)^{-q} \mathbf{1}_{\{\sup_{s\in[T_\beta+1,r]} y_s - y_{T_\beta+1} \le \beta - y_{T_\beta+1}\}}] \\ = & e^{-\frac{q\gamma(\beta-1)}{2}} \mathbb{E}[\mathbf{1}_{\{T_\beta+1 \le r\}} I(T_\beta)^{-q} \mathbb{E}[\mathbf{1}_{\{\sup_{s\in[0,r-T_\beta-1]} y'_s \le \beta - y_{T_\beta+1}\}} | \mathcal{F}_{T_\beta+1}]] \\ = & e^{-\frac{q\gamma(\beta-1)}{2}} \mathbb{E}\Big[\mathbf{1}_{\{T_\beta+1 \le r\}} \mathbb{E}[I(T_\beta)^{-q} | \mathcal{F}_{T_\beta}, \beta - y_{T_\beta+1}] \mathbb{E}[\mathbf{1}_{\{\sup_{s\in[0,r-T_\beta-1]} y'_s \le \beta - y_{T_\beta+1}\}} | \mathcal{F}_{T_\beta}, \beta - y_{T_\beta+1}]\Big]. \end{aligned}$$

By lemma 3.4.7,

$$\mathbb{E}[I(T_\beta)^{-q} | \mathcal{F}_{T_\beta}, \beta - y_{T_\beta+1}] \le C_1(e^{-\frac{\gamma}{2}q(y_{T_\beta+1}-\beta)} + 1) \quad a.s.$$

By lemma 3.4.6,

$$\mathbb{E}[\mathbf{1}_{\{\sup_{s\in[0,r-T_\beta-1]} y'_s\leq\beta-y_{T_\beta+1}\}}|\mathcal{F}_{T_\beta},\beta-y_{T_\beta+1}]\leq e^{\alpha(\beta-y_{T_\beta+1})-\frac{\alpha^2(r-T_\beta-1)}{2}}\quad a.s.$$

Therefore:

$$A\leq C_1e^{-\frac{q\gamma(\beta-1)}{2}}\mathbb{E}\Big[\mathbf{1}_{\{T_\beta+1\leq r\}}(e^{-\frac{\gamma}{2}q(y_{T_\beta+1}-\beta)}+1)e^{\alpha(\beta-y_{T_\beta+1})-\frac{\alpha^2(r-T_\beta-1)}{2}}\Big].$$

Conditioning on $\mathcal{F}_{T_\beta}$, $y_{T_\beta+1}-\beta$ has the law of $N+\alpha$ where $N\sim\mathcal{N}(0,1)$. Hence,

$$\begin{aligned}A\leq&C_1e^{-\frac{q\gamma(\beta-1)}{2}}\mathbb{E}[(e^{-\frac{\gamma}{2}q(N+\alpha)}+1)e^{-\alpha(N+\alpha)}]\mathbb{E}[\mathbf{1}_{\{T_\beta+1\leq r\}}e^{-\frac{\alpha^2(r-T_\beta-1)}{2}}]\\=&C_1e^{-\frac{q\gamma(\beta-1)}{2}}(e^{-\frac{\alpha^2}{2}+\frac{\gamma^2q^2}{8}}+e^{-\frac{\alpha^2}{2}})\mathbb{E}[\mathbf{1}_{\{T_\beta+1\leq r\}}e^{-\frac{\alpha^2(r-T_\beta-1)}{2}}]\\\leq&C_1e^{-\frac{q\gamma(\beta-1)}{2}}(e^{\frac{\gamma^2q^2}{8}}+1)e^{-\frac{\alpha^2r}{2}}\mathbb{E}[\mathbf{1}_{\{T_\beta\leq r-1\}}e^{\frac{\alpha^2T_\beta}{2}}].\end{aligned}$$

We calculate with the density of T_β:

$$\begin{aligned}\mathbb{E}[\mathbf{1}_{\{T_\beta\leq r-1\}}e^{\frac{\alpha^2T_\beta}{2}}]=&\int_0^{r-1}\frac{\beta-1}{\sqrt{2\pi}u^{3/2}}e^{-\frac{(\beta-1-\alpha u)^2}{2u}}e^{\frac{\alpha^2u}{2}}du=e^{\alpha(\beta-1)}\sqrt{\frac{2}{\pi}}\int_{\frac{\beta-1}{\sqrt{r-1}}}^{\infty}e^{-\frac{x^2}{2}}dx\\\leq&e^{\alpha(\beta-1)}\frac{1}{\sqrt{2\pi}}e^{-\frac{(\beta-1)^2}{2(r-1)}}.\end{aligned}$$

Combining the elements above we get:

$$A\leq C'_1e^{-\frac{\alpha^2r}{2}}e^{(\alpha-\frac{q\gamma}{2})\beta-\frac{(\beta-1)^2}{2r}}.\tag{3.4.13}$$

We proceed similarly for B:

$$\begin{aligned}B\leq&\mathbb{E}\Big[\mathbf{1}_{\{T_\beta>r-1\}}\mathbf{1}_{\{\beta-1-y_{r-1}\leq\sup_{0\leq s\leq1}y'_s\leq\beta-y_{r-1}\}}e^{-\frac{q\gamma y_{r-1}}{2}}\mathbb{E}[I(r-1)^{-q}|y_r-y_{r-1}]\Big]\\\leq&C_1\mathbb{E}[\mathbf{1}_{\{T_\beta>r-1\}}\mathbf{1}_{\{\beta-1-y_{r-1}\leq\sup_{0\leq s\leq1}y'_s\leq\beta-y_{r-1}\}}e^{-\frac{q\gamma(\beta-1-\sup_{0\leq s\leq1}y'_s)}{2}}(e^{-\frac{q\gamma}{2}(y_r-y_{r-1})}+1)]\\=&C_1(e^{\frac{q^2\gamma^2}{8}-\frac{q\gamma\alpha}{2}}+1)\mathbb{E}[e^{\frac{q\gamma}{2}\sup_{0\leq s\leq1}y'_s}]e^{-\frac{q\gamma(\beta-1)}{2}}\mathbb{E}[\mathbf{1}_{\{r-1<T_\beta\leq r\}}].\end{aligned}$$

We then compute:

$$\mathbb{E}[\mathbf{1}_{\{r-1<T_\beta\leq r\}}]=\int_{r-1}^{r}\frac{\beta-1}{\sqrt{2\pi}u^{3/2}}e^{-\frac{(\beta-1-\alpha u)^2}{2u}}du\leq\frac{\beta-1}{\sqrt{2\pi}(r-1)^{3/2}}e^{\alpha(\beta-1)-\frac{(\beta-1)^2}{2r}-\frac{\alpha^2(r-1)}{2}}du.$$

Therefore:

$$B\leq C''_1\frac{e^{-\frac{\alpha^2r}{2}}}{(r-1)^{3/2}}(\beta-1)e^{(\alpha-\frac{q\gamma}{2})\beta-\frac{(\beta-1)^2}{2r}}.\tag{3.4.14}$$

Equations (3.4.13) and (3.4.14) together finish the proof of the lemma. □

Now we can prove the main lemma:

Proof of lemma 3.4.5. Define for $n \geq 1$:

$$M_n = \{ \sup_{s\in[r_0,r]} (y_s - y_{r_0}) \in [n-1, n]\}. \tag{3.4.15}$$

We can write,

$$\begin{aligned}\mathbb{E}[(\int_{r_0}^{r} e^{\frac{\gamma}{2}y_s}\mu_Y(ds))^{-q}] =& e^{(\frac{q^2\gamma^2}{8}-\frac{q\gamma\alpha}{2})r_0}\sum_{n\geq 1}\mathbb{E}[\mathbf{1}_{M_n}(\int_{r_0}^{r} e^{\frac{\gamma}{2}(y_s-y_{r_0})}\mu_Y(ds))^{-q}]\\ =& e^{(\frac{q^2\gamma^2}{8}-\frac{q\gamma\alpha}{2})r_0}\sum_{n\geq 1} J_{r-r_0,n},\end{aligned}$$

and by lemma 3.4.8:

$$J_{r-r_0,n} \leq C_2 e^{-\frac{\alpha^2(r-r_0)}{2}} e^{(\alpha-\frac{q\gamma}{2})n-\frac{(n-1)^2}{2(r-r_0)}}(1+\frac{n-1}{(r-r_0-1)^{3/2}}).$$

Remark that it is straight forward if $\alpha - \frac{q\gamma}{2} < 0$, but we can have the opposite sign in our case, so we need to take into consideration the term $e^{-\frac{(n-1)^2}{2(r-r_0)}}$. By a comparison of series and integrals, we can show that there exists a sufficiently large $r_1 > r_0$ such that for all $r \geq r_1$,

$$\sum_{n\geq 1} e^{(\alpha-\frac{q\gamma}{2})n-\frac{(n-1)^2}{2(r-r_0)}}(1+\frac{n-1}{(r-r_0-1)^{3/2}}) \leq C_3\sqrt{r},$$

where $C_3 > 0$ is independent of r. Finally:

$$\mathbb{E}[(\int_{r_0}^{r} e^{\frac{\gamma}{2}y_s}\mu_Y(ds))^{-q}] \leq C\sqrt{r}e^{-\frac{\alpha^2 r}{2}}.$$

□

3.4.3 Fusion estimation and the reflection coefficient

In this subsection we discuss a second way to develop $U(t)$ (the first is by Taylor expansion), which will give us the shift equation (3.1.17) on p with a shift $\frac{4}{\gamma^2}$. In this expansion will appear the reflection coefficient introduced in section 3.1.2.

Lemma 3.4.9. *For $-1-\frac{\gamma^2}{4} < a < -1-\frac{\gamma^2}{8}$, $p < 1+\frac{4}{\gamma^2}(a+1)$, as $t \to 0_-$,*

$$\begin{aligned}U(t) =& M(\gamma,p,a+\frac{\gamma^2}{4},0) + f(\gamma,a)\frac{\Gamma(-p+1+\frac{4}{\gamma^2}(a+1))}{\Gamma(-p)}|t|^{1+a+\frac{\gamma^2}{4}}M(\gamma,p-1-\frac{4}{\gamma^2}(a+1),-2-a-\frac{\gamma^2}{4},0)\\ &+ o(|t|^{1+a+\frac{\gamma^2}{4}}),\end{aligned} \tag{3.4.16}$$

where $f(\gamma,a)$ is defined as:

$$f(\gamma,a) = -\Gamma(-\frac{4}{\gamma^2}(a+1))\mathbb{E}[(\frac{1}{2}\int_{-\infty}^{\infty} e^{\frac{\gamma}{2}\mathcal{B}_s^{\frac{\gamma}{4}+\frac{1}{\gamma}(a+1)}}\mu_Y(ds))^{1+\frac{4}{\gamma^2}(a+1)}]. \tag{3.4.17}$$

The process $\mathcal{B}^{\frac{\gamma}{4}+\frac{1}{\gamma}(a+1)}$ is defined by (3.1.26) and $\mu_Y(ds) = e^{\frac{\gamma}{2}Y(e^{-s/2})}ds$ is the notation introduced in section 3.4.2.

Notice that in the expression of $f(\gamma, a)$ we recognize the reflection coefficient $\overline{R}_1^{\partial}(-\frac{2a}{\gamma})$ of section 3.1.2. We also mention that the bound $a < -1 - \frac{\gamma^2}{8}$ is not optimal but it is sufficient for our purposes.

Remark 3.4.10. *From the conditions on a and p in the lemma, we have $-2-a-\frac{\gamma^2}{4} > -1-\frac{\gamma^2}{4}$ and $p-1-\frac{4}{\gamma^2}(a+1) < 0$, thus the bounds (3.1.5) are satisfied and $M(\gamma, p-1-\frac{4}{\gamma^2}(a+1), -2-a-\frac{\gamma^2}{4}, 0)$ is well defined. We also want to mention that a similar result holds for $\tilde{U}(t)$ and the proof is almost the same.*

Proof. We adapt the arguments in [47] for the proof of this lemma. We introduce the notation

$$K_I(t) := \int_I (x-t)^{\frac{\gamma^2}{4}} x^a e^{\frac{\gamma}{2}X(x)} dx \tag{3.4.18}$$

for a borel set $I \subseteq [0,1]$. Recall that we work with $-1-\frac{\gamma^2}{4} < a < -1-\frac{\gamma^2}{8}$, hence $p < 1+\frac{4}{\gamma^2}(a+1) < 1$. We want to study the asymptotic of

$$\mathbb{E}[K_{[0,1]}(t)^p] - \mathbb{E}[K_{[0,1]}(0)^p] =: T_1 + T_2, \tag{3.4.19}$$

where we defined:

$$T_1 := \mathbb{E}[K_{[|t|,1]}(t)^p] - \mathbb{E}[K_{[0,1]}(0)^p], \quad T_2 := \mathbb{E}[K_{[0,1]}(t)^p] - \mathbb{E}[K_{[|t|,1]}(t)^p]. \tag{3.4.20}$$

$\diamond$ First we consider T_1. The goal is to show that $T_1 = o(|t|^{1+a+\frac{\gamma^2}{4}})$. By interpolation,

$$\begin{aligned}|T_1| \leq &|p| \int_0^1 du \mathbb{E}[|K_{[|t|,1]}(t) - K_{[0,1]}(0)|(uK_{[|t|,1]}(t) + (1-u)K_{[0,1]}(0))^{p-1}] \\ \leq &|p|\mathbb{E}[|K_{[|t|,1]}(t) - K_{[0,1]}(0)|K_{[|t|,1]}(0)^{p-1}] \leq p(A_1 + A_2),\end{aligned} \tag{3.4.21}$$

where $A_1 = \mathbb{E}[|K_{[|t|,1]}(t) - K_{[|t|,1]}(0)|K_{[|t|,1]}(0)^{p-1}]$ and $A_2 = \mathbb{E}[|K_{[|t|,1]}(0) - K_{[0,1]}(0)|K_{[|t|,1]}(0)^{p-1}]$. We start by estimating A_1. Using the sub-additivity of the function $x \mapsto x^{\frac{\gamma^2}{4}}$,

$$\begin{aligned}A_1 =&\mathbb{E}[|K_{[|t|,1]}(t) - K_{[|t|,1]}(0)|K_{[|t|,1]}(0)^{p-1}] \leq |t|^{\frac{\gamma^2}{4}} \int_{|t|}^1 dx_1\, x_1^a \mathbb{E}[(\int_{|t|}^1 \frac{x^{a+\frac{\gamma^2}{4}}}{|x-x_1|^{\frac{\gamma^2}{2}}} e^{\frac{\gamma}{2}X(x)} dx)^{p-1}] \\ \leq &|t|^{\frac{\gamma^2}{4}} \int_{|t|}^{t_0} dx_1\, x_1^a \mathbb{E}[(\int_{x_1}^1 x^{a-\frac{\gamma^2}{4}} e^{\frac{\gamma}{2}X(x)} dx)^{p-1}] + C|t|^{\frac{\gamma^2}{4}},\end{aligned}$$

where t_0 is a constant in $(0,1)$ to be fixed. Note that in this subsection we will use C to denote all the constants. We use lemma 3.4.4, there exists $\epsilon_1 > 0$ such that for all $x_1 < \epsilon_1$:

$$\mathbb{E}[(\int_{x_1}^1 x^{a-\frac{\gamma^2}{4}} e^{\frac{\gamma}{2}X(x)} dx)^{p-1}] \leq C\sqrt{\ln(1/x_1)}\, x_1^{\frac{1}{\gamma^2}(a+1)^2}. \tag{3.4.22}$$

Taking $t_0 = \epsilon_1$ we obtain:

$$\begin{aligned}A_1 \leq &C|t|^{\frac{\gamma^2}{4}} \int_{|t|}^{\epsilon_1} dx_1 \sqrt{\ln(1/x_1)}\, x_1^{a+\frac{1}{\gamma^2}(a+1)^2} + C|t|^{\frac{\gamma^2}{4}} \\ \leq &C\sqrt{\ln(1/|t|)}\, |t|^{1+\frac{\gamma^2}{4}+a+\frac{1}{\gamma^2}(a+1)^2} + C|t|^{\frac{\gamma^2}{4}} = o\big(|t|^{1+a+\frac{\gamma^2}{4}}\big).\end{aligned} \tag{3.4.23}$$

On the other hand:

$$A_2 = \mathbb{E}[K_{[0,|t|]}(0)K_{[|t|,1]}(0)^{p-1}] = \int_0^{|t|} dx_1\, x_1^{a+\frac{\gamma^2}{4}} \mathbb{E}[(\int_{|t|}^1 \frac{x^{a+\frac{\gamma^2}{4}}}{|x-x_1|^{\frac{\gamma^2}{2}}} e^{\frac{\gamma}{2}X(x)} dx)^{p-1}]$$
$$\leq \int_0^{|t|} dx_1\, x_1^{a+\frac{\gamma^2}{4}} \mathbb{E}[(\int_{|t|}^1 x^{a-\frac{\gamma^2}{4}} e^{\frac{\gamma}{2}X(x)} dx)^{p-1}]$$
$$\overset{(3.4.22)}{\leq} C\sqrt{\ln(1/|t|)}\, |t|^{1+a+\frac{\gamma^2}{4}+\frac{1}{\gamma^2}(a+1)^2} = o(|t|^{1+a+\frac{\gamma^2}{4}}). \tag{3.4.24}$$

Hence we have shown that $T_1 = o(|t|^{1+a+\frac{\gamma^2}{4}})$.

◇ Now we focus on T_2. The goal is to restrict K to the complementary of $[|t|^2, |t|]$ and then on the two parts the GMC's are weakly correlated. Remark that the choice of $|t|^2$ can be ameliorated to get a larger bound for a instead of $a < -1 - \frac{\gamma^2}{8}$. The same computation as (3.4.21) shows that:

$$\mathbb{E}[K_{[0,1]}(t)^p] - \mathbb{E}[K_{[|t|^2,|t|]^c}(t)^p] \leq |p|\mathbb{E}[K_{[0,|t|]}(t)K_{[|t|,1]}(0)^{p-1}]$$
$$\leq |p|\mathbb{E}[|K_{[0,|t|]}(t) - K_{[0,|t|]}(0)|K_{[|t|,1]}(0)^{p-1}] + |p|A_2.$$

The first term can be bounded similarly as A_1 by $o(|t|^{1+a+\frac{\gamma^2}{4}})$, and A_2 also has been proved to be $o(|t|^{1+a+\frac{\gamma^2}{4}})$, hence

$$\mathbb{E}[K_{[0,1]}(t)^p] - \mathbb{E}[K_{[|t|^2,|t|]^c}(t)^p] = o(|t|^{1+a+\frac{\gamma^2}{4}}) \tag{3.4.25}$$

This means that it suffices to evaluate $\mathbb{E}[K_{[|t|^2,|t|]^c}(t)^p] - \mathbb{E}[K_{[|t|,1]}(t)^p]$. We will use the radial decomposition of X with the notations introduced in the first paragraph of section 3.4.2,

$$K_1(t) := K_{[|t|,1]}(t) = \frac{1}{2}\int_0^{2\ln\frac{1}{|t|}} (e^{-s/2} - t)^{\frac{\gamma^2}{4}} e^{\frac{\gamma}{2}(B_s - s(\frac{\gamma}{4} + \frac{1}{\gamma}(a+1)))} \mu_Y(ds), \tag{3.4.26}$$

$$K_2(t) := K_{[0,|t|^2]}(t) = \frac{1}{2}\int_{4\ln\frac{1}{|t|}}^{\infty} (e^{-s/2} - t)^{\frac{\gamma^2}{4}} e^{\frac{\gamma}{2}(B_s - s(\frac{\gamma}{4} + \frac{1}{\gamma}(a+1)))} \mu_Y(ds). \tag{3.4.27}$$

From (3.4.4), we deduce that for $s \leq 2\ln\frac{1}{|t|}$ and $s' \geq 4\ln\frac{1}{|t|}$,

$$0 \leq \mathbb{E}[Y(e^{-s/2})Y(e^{-s'/2})] = \ln\frac{1}{|1 - e^{-(s'-s)/2}|} \leq 2|t|, \tag{3.4.28}$$

where we used the inequality $\ln\frac{1}{1-x} \leq 2x$ for $x \in [0, \frac{1}{2}]$. Define the process,

$$P(e^{-s/2}) := Y(e^{-s/2})\mathbf{1}_{\{s \leq 2\ln\frac{1}{|t|}\}} + Y(e^{-s/2})\mathbf{1}_{\{s \geq 4\ln\frac{1}{|t|}\}},$$

$$\tilde{P}(e^{-s/2}) := Y(e^{-s/2})\mathbf{1}_{\{s \leq 2\ln\frac{1}{|t|}\}} + \tilde{Y}(e^{-s/2})\mathbf{1}_{\{s \geq 4\ln\frac{1}{|t|}\}},$$

where $\tilde{Y}$ is a gaussian field independent from everything and has the same law as Y. Then we have the inequality over the covariance:

$$\mathbb{E}[\tilde{P}(e^{-s/2})\tilde{P}(e^{-s'/2})] \leq \mathbb{E}[P(e^{-s/2})P(e^{-s'/2})] \leq \mathbb{E}[\tilde{P}(e^{-s/2})\tilde{P}(e^{-s'/2})] + 4|t|. \tag{3.4.29}$$

The function $x \mapsto x^p$ is convex when $p \leq 0$ and concave when $0 < p < 1$. We will only work with the case $p \leq 0$ since the case $0 < p < 1$ can be treated in the same way. By applying Kahane's inequality of Theorem 3.4.2,

$$\mathbb{E}[(K_1(t) + \tilde{K}_2(t))^p] \leq \mathbb{E}[(K_1(t) + K_2(t))^p] \leq e^{\frac{\gamma^2}{2}(p^2-p)|t|}\mathbb{E}[(K_1(t) + \tilde{K}_2(t))^p], \tag{3.4.30}$$

where $\tilde{K}_2(t) := \frac{1}{2}\int_{4\ln\frac{1}{|t|}}^{\infty}(e^{-s/2} - t)^{\frac{\gamma^2}{4}} e^{\frac{\gamma}{2}(B_s - s(\frac{\gamma}{4}+\frac{1}{\gamma}(a+1)))}\mu_{\tilde{Y}}(ds)$. By the Markov property of Brownian motion and stationarity of $\mu_{\tilde{Y}}$, we have

$$\tilde{K}_2(t) := \frac{1}{2}|t|^{2+2a+\frac{3\gamma^2}{4}}e^{\frac{\gamma}{2}B_{4\ln(1/|t|)}}\int_0^{\infty}(|t|e^{-s/2} + 1)^{\frac{\gamma^2}{4}} e^{\frac{\gamma}{2}(\tilde{B}_s - s(\frac{\gamma}{4}+\frac{1}{\gamma}(a+1)))}\mu_{\tilde{Y}}(ds), \tag{3.4.31}$$

with $\tilde{B}$ an independent Brownian motion. We denote

$$\sigma_t := |t|^{2+2a+\frac{3\gamma^2}{4}}e^{\frac{\gamma}{2}B_{4\ln(1/|t|)}}, \quad V := \frac{1}{2}\int_0^{\infty} e^{\frac{\gamma}{2}(\tilde{B}_s - s(\frac{\gamma}{4}+\frac{1}{\gamma}(a+1)))}\mu_{\tilde{Y}}(ds), \tag{3.4.32}$$

then:

$$\mathbb{E}[(K_1(t) + (1+|t|)^{\frac{\gamma^2}{4}}\sigma_t V)^p] \leq \mathbb{E}[(K_1(t) + K_2(t))^p] \leq e^{\frac{\gamma^2}{2}(p^2-p)|t|}\mathbb{E}[(K_1(t) + \sigma_t V)^p]. \tag{3.4.33}$$

By the Williams path decomposition of Theorem 3.4.3 we can write,

$$V = e^{\frac{\gamma}{2}M}\frac{1}{2}\int_{-L_M}^{\infty} e^{\frac{\gamma}{2}\mathcal{B}_s^{\lambda}}\mu_{\tilde{Y}}(ds), \tag{3.4.34}$$

where $\lambda = \frac{\gamma}{4} + \frac{1}{\gamma}(a+1)$, $M = \sup_{s\geqslant 0}(\tilde{B}_s - \lambda s)$ and L_M is the last time $\mathcal{B}_s^{\lambda}$ hits $-M$. Recall that the law of M is known, for $v \geq 1$,

$$\mathbb{P}(e^{\frac{\gamma}{2}M} > v) = \frac{1}{v^{\frac{4\lambda}{\gamma}}}. \tag{3.4.35}$$

For simplicity, we introduce the notations:

$$\rho_A(\lambda) := \frac{1}{2}\int_{-L_A}^{\infty} e^{\frac{\gamma}{2}\mathcal{B}_s^{\lambda}}\mu_{\tilde{Y}}(ds), \quad \rho(\lambda) := \frac{1}{2}\int_{-\infty}^{\infty} e^{\frac{\gamma}{2}\mathcal{B}_s^{\lambda}}\mu_{\tilde{Y}}(ds). \tag{3.4.36}$$

Now we discuss the lower and upper bound separately.

◇ Lower bound: Since we work with $p \leq 0$,

$$\begin{aligned}
&\mathbb{E}[(K_1(t) + K_2(t))^p] - \mathbb{E}[K_1(t)^p]\\
&\geq \mathbb{E}[(K_1(t) + (1+|t|)^{\frac{\gamma^2}{4}}\sigma_t e^{\frac{\gamma}{2}M}\rho(\lambda))^p] - \mathbb{E}[K_1(t)^p]\\
&= \frac{4\lambda}{\gamma}\mathbb{E}\Big[\int_1^{\infty}\frac{dv}{v^{\frac{4\lambda}{\gamma}+1}}\Big((K_1(t) + (1+|t|)^{\frac{\gamma^2}{4}}\sigma_t\rho(\lambda)v)^p - K_1(t)^p\Big)\Big]\\
&= \frac{4\lambda}{\gamma}\mathbb{E}\Big[\int_{\frac{(1+|t|)^{\frac{\gamma^2}{4}}\sigma_t\rho(\lambda)}{K_1(t)}}^{\infty}\frac{du}{u^{\frac{4\lambda}{\gamma}+1}}((u+1)^p - 1)((1+|t|)^{\frac{\gamma^2}{4}}\sigma_t\rho(\lambda))^{\frac{4\lambda}{\gamma}}K_1(t)^{p-\frac{4\lambda}{\gamma}}\Big]\\
&\overset{(3.4.58)}{\geq} \frac{4\lambda}{\gamma}\frac{\Gamma(-p+\frac{4\lambda}{\gamma})\Gamma(-\frac{4\lambda}{\gamma})}{\Gamma(-p)}\mathbb{E}[((1+|t|)^{\frac{\gamma^2}{4}}\sigma_t\rho(\lambda))^{\frac{4\lambda}{\gamma}}K_1(t)^{p-\frac{4\lambda}{\gamma}}].
\end{aligned}$$

By the Girsanov theorem,

$$\begin{aligned}
&\mathbb{E}[((1+|t|)^{\frac{\gamma^2}{4}}\sigma_t\rho(\lambda))^{\frac{4\lambda}{\gamma}}K_1(t)^{p-\frac{4\lambda}{\gamma}}] \\
&\quad = \big(|t|(1+|t|)\big)^{1+a+\frac{\gamma^2}{4}}\mathbb{E}[\rho(\lambda)^{\frac{4\lambda}{\gamma}}]\mathbb{E}\big[\big(\frac{1}{2}\int_0^{2\ln\frac{1}{|t|}}(e^{-s/2}-t)^{\frac{\gamma^2}{4}}e^{\frac{\gamma}{2}(B_s+s(\frac{\gamma}{4}+\frac{1}{\gamma}(a+1)))}\mu_Y(ds)\big)^{p-\frac{4\lambda}{\gamma}}\big] \\
&\quad \underset{t\to 0_-}{\sim} |t|^{1+a+\frac{\gamma^2}{4}}\mathbb{E}[\rho(\lambda)^{\frac{4\lambda}{\gamma}}]M(\gamma,p-1-\frac{4}{\gamma^2}(a+1),-2-a-\frac{\gamma^2}{4},0). \qquad (3.4.37)
\end{aligned}$$

This completes the proof for lower bound.

◇ Upper bound: we start with an inequality:

$$\begin{aligned}
&\mathbb{E}[\big((K_1(t)+K_2(t))^p] - \mathbb{E}[K_1(t)^p] \\
&\quad \le \mathbb{E}[(K_1(t)+\sigma_t V)^p] - \mathbb{E}[K_1(t)^p] + (e^{\frac{\gamma^2}{2}(p^2-p)|t|}-1)\mathbb{E}[K_1(0)^p] \qquad (3.4.38) \\
&\quad = \mathbb{E}[(K_1(t)+\sigma_t V)^p] - \mathbb{E}[K_1(t)^p] + O(|t|). \qquad (3.4.39)
\end{aligned}$$

For $A>0$ fixed, since $p\le 0$ we have,

$$\begin{aligned}
&\mathbb{E}[(K_1(t)+\sigma_t V)^p - K_1(t)^p] \le \mathbb{E}[\big((K_1(t)+\sigma_t V)^p - K_1(t)^p\big)\mathbf{1}_{\{M>A\}}] \\
&\quad \le \mathbb{E}[\big((K_1(t)+\sigma_t\, e^{\frac{\gamma}{2}M}\rho_A(\lambda))^p - K_1(t)^p\big)\mathbf{1}_{\{M>A\}}] \\
&\quad \overset{(3.4.35)}{=} \frac{4\lambda}{\gamma}\mathbb{E}\Big[\int_{\frac{e^{\gamma A/2}\sigma_t\rho_A(\lambda)}{K_1(t)}}^{\infty}\frac{du}{u^{\frac{4\lambda}{\gamma}+1}}((u+1)^p-1)(\sigma_t\rho_A(\lambda))^{\frac{4\lambda}{\gamma}}K_1(t)^{p-\frac{4\lambda}{\gamma}}\Big] \\
&\quad \overset{\text{Girsanov}}{=} \frac{4\lambda}{\gamma}|t|^{1+a+\frac{\gamma^2}{4}}\mathbb{E}\Big[\int_{\frac{e^{\gamma A/2}\hat{\sigma}_t\rho_A(\lambda)}{\hat{K}_1(t)}}^{\infty}\frac{du}{u^{\frac{4\lambda}{\gamma}+1}}((u+1)^p-1)\rho_A(\lambda)^{\frac{4\lambda}{\gamma}}\hat{K}_1(t)^{p-\frac{4\lambda}{\gamma}}\Big],
\end{aligned}$$

where

$$\begin{aligned}
\hat{K}_1(t) &= \mathbb{E}\big[\big(\frac{1}{2}\int_0^{2\ln\frac{1}{|t|}}(e^{-s/2}-t)^{\frac{\gamma^2}{4}}e^{\frac{\gamma}{2}(B_s+s(\frac{\gamma}{4}+\frac{1}{\gamma}(a+1)))}\mu_Y(ds)\big)^{p-\frac{4\lambda}{\gamma}}\big] \\
&\overset{t\to 0_-}{\sim} M(\gamma,p-1-\frac{4}{\gamma^2}(a+1),-2-a-\frac{\gamma^2}{4},0),
\end{aligned}$$

and for $a<-1-\frac{\gamma^2}{8}$,

$$\hat{\sigma_t} = |t|^{-2-2a-\frac{\gamma^2}{4}}e^{\frac{\gamma}{2}B_{4\ln(1/|t|)}} \overset{t\to 0_-}{\longrightarrow} 0 \quad \text{a.s.}$$

Hence $\mathbb{E}[(K_1(t)+\sigma_t V)^p - K_1(t)^p]$ is smaller than a term equivalent to:

$$\frac{4\lambda}{\gamma}\frac{\Gamma(-p+\frac{4\lambda}{\gamma})\Gamma(-\frac{4\lambda}{\gamma})}{\Gamma(-p)}|t|^{1+a+\frac{\gamma^2}{4}}\mathbb{E}[\rho_A(\lambda)^{\frac{4\lambda}{\gamma}}]M(\gamma,p-1-\frac{4}{\gamma^2}(a+1),-2-a-\frac{\gamma^2}{4},0).$$

We can conclude by sending A to ∞. □

3.4.4 Computation of the reflection coefficient

The goal of this subsection is to prove Proposition 3.1.5. In the first step we give a proof of the tail expansion (3.1.27) where the coefficient $\overline{R}_1^\partial$ is expressed in terms of the processes Y and $\mathcal{B}_s^\alpha$ as defined in the section 3.1.2. The proof is almost the same as in [47]. In the second step we provide the exact value (3.1.28) for $\overline{R}_1^\partial$ by using Theorem 3.1.1. Before proving the proposition, we provide a useful lemma. The proof can be found in [47] (see lemma 2.8).

Lemma 3.4.11. *Let $\alpha \in (\frac{\gamma}{2}, Q)$ with $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$, then for $p < \frac{4}{\gamma^2}$ and all non trivial interval $I \subseteq \mathbb{R}$:*

$$\mathbb{E}[(\frac{1}{2}\int_I e^{\frac{\gamma}{2}\mathcal{B}_s^{\frac{Q-\alpha}{2}}} e^{\frac{\gamma}{2}Y(e^{-s/2})}ds)^p] < \infty. \tag{3.4.40}$$

This lemma tells us that the additional term $e^{\frac{\gamma}{2}\mathcal{B}_s^{\frac{Q-\alpha}{2}}}$ behaves nicely and the bound on p is the same as in the case of GMC moments.

Proof of Proposition 3.1.5. Using the decomposition $X(e^{-s/2}) = B_s + Y(e^{-s/2})$ we have,

$$\begin{aligned} I_{1,\eta}^\partial(\alpha) &= \int_0^\eta x^{-\frac{\gamma\alpha}{2}} e^{\frac{\gamma}{2}X(x)}dx = \frac{1}{2}\int_{-2\ln\eta}^\infty e^{\frac{\gamma}{2}(B_s - s(\frac{\gamma}{4}+\frac{1}{\gamma}-\frac{\alpha}{2}))} e^{\frac{\gamma}{2}Y(e^{-s/2})}ds \\ &\overset{\text{Theorem 3.4.3}}{=} e^{\frac{\gamma}{2}M}\frac{1}{2}\int_{-2\ln\eta - L_M}^\infty e^{\frac{\gamma}{2}\mathcal{B}_s^{\frac{Q-\alpha}{2}}} e^{\frac{\gamma}{2}Y(e^{-s/2})}ds, \end{aligned}$$

where $M = \sup_{s\geqslant 0}(\tilde{B}_s - \frac{Q-\alpha}{2}s)$ and L_M is the last time $\mathcal{B}_s^{\frac{Q-\alpha}{2}}$ hits $-M$. The law of M is given by:

$$\mathbb{P}(e^{\frac{\gamma}{2}M} > v) = \frac{1}{v^{\frac{2(Q-\alpha)}{\gamma}}}. \tag{3.4.41}$$

We denote

$$\rho_A(\frac{Q-\alpha}{2}) = \frac{1}{2}\int_{-L_A}^\infty e^{\frac{\gamma}{2}\mathcal{B}_s^{\frac{Q-\alpha}{2}}} e^{\frac{\gamma}{2}Y(e^{-s/2})}ds, \quad \rho(\frac{Q-\alpha}{2}) = \frac{1}{2}\int_{-\infty}^\infty e^{\frac{\gamma}{2}\mathcal{B}_s^{\frac{Q-\alpha}{2}}} e^{\frac{\gamma}{2}Y(e^{-s/2})}ds,$$

and study the upper and lower bounds for $\mathbb{P}(I_{1,\eta}^\partial(\alpha) > u)$:

◇ Upper bound: $\mathbb{P}(I_{1,\eta}^\partial(\alpha) > u) \le \mathbb{P}(e^{\frac{\gamma}{2}M}\rho(\frac{Q-\alpha}{2}) > u) = \frac{\mathbb{E}[\rho(\frac{Q-\alpha}{2})^{\frac{2(Q-\alpha)}{\gamma}}]}{u^{\frac{2(Q-\alpha)}{\gamma}}} = \frac{\overline{R}_1^\partial(\alpha)}{u^{\frac{2}{\gamma}(Q-\alpha)}}$.

◇ Lower bound: we first show that the tail behavior is concentrated at $x = 0$ and that the value of η does not matter. Consider $h, \epsilon > 0$ sufficiently small,

$$\begin{aligned} \mathbb{P}(I_{1,1}^\partial(\alpha) > u + u^{1-h}) - \mathbb{P}(I_{1,\eta}^\partial(\alpha) > u) &\le \mathbb{P}(\int_\eta^1 x^{-\frac{\gamma\alpha}{2}} e^{\frac{\gamma}{2}X(x)}dx > u^{1-h}) \\ &\le \frac{\mathbb{E}[(\int_\eta^1 x^{-\frac{\gamma\alpha}{2}} e^{\frac{\gamma}{2}X(x)}dx)^{\frac{4}{\gamma^2}-\epsilon}]}{u^{(1-h)(\frac{4}{\gamma^2}-\epsilon)}} = O_{u\to\infty}(\frac{1}{u^{\frac{2(Q-\alpha)}{\gamma}+\nu}}), \end{aligned} \tag{3.4.42}$$

where ν is some constant in $(0, 1)$. Thus it suffices to study the tail behavior of $I_{1,1}^\partial(\alpha)$. Take $A = \frac{2\nu}{\gamma}\ln u$,

$$\mathbb{P}(I_{1,1}^\partial(\alpha) > u) \ge \mathbb{P}(e^{\frac{\gamma}{2}M}\rho_A(\frac{Q-\alpha}{2}) > u, M > A) = \min\{\frac{\mathbb{E}[\rho_A(\frac{Q-\alpha}{2})^{\frac{2(Q-\alpha)}{\gamma}}]}{u^{\frac{2(Q-\alpha)}{\gamma}}}, e^{-(Q-\alpha)A}\}.$$

Notice that $u^{-\frac{2(Q-\alpha)}{\gamma}} = o_{u\to\infty}(e^{-(Q-\alpha)A})$, hence for u sufficiently large:

$$\mathbb{P}(I^{\partial}_{1,\eta}(\alpha) > u) - \frac{\overline{R}^{\partial}_1(\alpha)}{u^{\frac{2(Q-\alpha)}{\gamma}}} \geq -\frac{\mathbb{E}[\rho(\frac{Q-\alpha}{2})^{\frac{2(Q-\alpha)}{\gamma}}] - \mathbb{E}[\rho_A(\frac{Q-\alpha}{2})^{\frac{2(Q-\alpha)}{\gamma}}]}{u^{\frac{2(Q-\alpha)}{\gamma}}}.$$

We claim that for $u > 1$,

$$\mathbb{E}[\rho(\frac{Q-\alpha}{2})^{\frac{2(Q-\alpha)}{\gamma}}] - \mathbb{E}[\rho_A(\frac{Q-\alpha}{2})^{\frac{2(Q-\alpha)}{\gamma}}] \leq Cu^{-\nu}. \tag{3.4.43}$$

This shows that:

$$\mathbb{P}(I^{\partial}_{1,1}(\alpha) > u) = \frac{\overline{R}^{\partial}_1(\alpha)}{u^{\frac{2(Q-\alpha)}{\gamma}}} + O(\frac{1}{u^{\frac{2(Q-\alpha)}{\gamma}+\nu}}). \tag{3.4.44}$$

By applying the tail result to (3.4.42) we deduce

$$\mathbb{P}(I^{\partial}_{1,\eta}(\alpha) > u) = \frac{\overline{R}^{\partial}_1(\alpha)}{u^{\frac{2(Q-\alpha)}{\gamma}}} + O(\frac{1}{u^{\frac{2(Q-\alpha)}{\gamma}+\min(\nu,h)}}), \tag{3.4.45}$$

which finishes the proof for the first part. For the second part let $\epsilon > 0$, the value of $\overline{R}^{\partial}_1(\alpha)$ is then determined by the following limit, with $p = \frac{2(Q-\alpha)}{\gamma}$,

$$\lim_{\epsilon\to 0} \epsilon\mathbb{E}[I^{\partial}_{1,1}(\alpha)^{p-\epsilon}] = p\overline{R}^{\partial}_1(\alpha). \tag{3.4.46}$$

With our Theorem 3.1.1 we can compute this limit and get:

$$\begin{aligned}
p\overline{R}^{\partial}_1(\alpha) &= \frac{(2\pi)^p(\frac{2}{\gamma})^{p\frac{\gamma^2}{4}}\Gamma_\gamma(\frac{2}{\gamma}-p\frac{\gamma}{2})\Gamma_\gamma(\frac{2}{\gamma}-(p-1)\frac{\gamma}{2})\Gamma_\gamma(\frac{4}{\gamma}-\alpha-(p-2)\frac{\gamma}{2})}{\Gamma(1-\frac{\gamma^2}{4})^p\Gamma_\gamma(\frac{2}{\gamma})\Gamma_\gamma(\frac{2}{\gamma}-\alpha+\frac{\gamma}{2})\Gamma_\gamma(\frac{2}{\gamma}+\frac{\gamma}{2})\Gamma_\gamma(\frac{4}{\gamma}-\alpha-(2p-2)\frac{\gamma}{2})} \lim_{\epsilon\to 0}\epsilon\Gamma_\gamma(\frac{\gamma\epsilon}{2}) \\
&= \frac{(2\pi)^p(\frac{2}{\gamma})^{p\frac{\gamma^2}{4}}\Gamma_\gamma(\alpha-\frac{\gamma}{2})\Gamma_\gamma(\frac{\gamma}{2})}{\Gamma(1-\frac{\gamma^2}{4})^p\Gamma_\gamma(\frac{2}{\gamma})\Gamma_\gamma(Q-\alpha)} \frac{1}{\sqrt{2\pi}}(\frac{\gamma}{2})^{-\frac{1}{2}}\Gamma_\gamma(\frac{2}{\gamma}) \\
&= \frac{1}{\sqrt{\gamma\pi}} \frac{(2\pi)^{\frac{2}{\gamma}(Q-\alpha)}(\frac{2}{\gamma})^{\frac{\gamma}{2}(Q-\alpha)}}{\Gamma(1-\frac{\gamma^2}{4})^{\frac{2}{\gamma}(Q-\alpha)}} \frac{\Gamma_\gamma(\alpha-\frac{\gamma}{2})}{\Gamma_\gamma(Q-\alpha)}.
\end{aligned}$$

It remains to show (3.4.43). By (3.4.3) of the Williams decomposition theorem of appendix 3.4.1, the process $\hat{\mathcal{B}}_s^{\frac{Q-\alpha}{2}}$ defined for $s \leq 0$ by $\hat{\mathcal{B}}_s^{\frac{Q-\alpha}{2}} = \mathcal{B}^{\frac{Q-\alpha}{2}}_{s-L_{\frac{2\nu}{\gamma}\ln u}} + \frac{2\nu}{\gamma}\ln u$ is independent from everything and has the same law as $(\mathcal{B}_s^{\frac{Q-\alpha}{2}})_{s\leq 0}$. We can then write,

$$\rho(\frac{Q-\alpha}{2}) = A_1 + u^{-\nu}A_2, \tag{3.4.47}$$

where:

$$A_1 = \frac{1}{2}\int_{-L_{\frac{2\nu}{\gamma}\ln u}}^{\infty} e^{\frac{\gamma}{2}\mathcal{B}_s^{\frac{Q-\alpha}{2}}} e^{\frac{\gamma}{2}Y(e^{-s/2})}ds, \quad A_2 = \frac{1}{2}\int_{-\infty}^{0} e^{\frac{\gamma}{2}\hat{\mathcal{B}}_s^{\frac{Q-\alpha}{2}}} e^{\frac{\gamma}{2}Y(e^{-s/2})}ds. \tag{3.4.48}$$

By interpolation (see (3.4.21) for example),

$$\mathbb{E}[(A_1+u^{-\nu}A_2)^{\frac{2(Q-\alpha)}{\gamma}}-A_1^{\frac{2(Q-\alpha)}{\gamma}}]\leq\frac{2(Q-\alpha)}{\gamma}u^{-\nu}\mathbb{E}[A_2\max\{\rho(\frac{Q-\alpha}{2})^{\frac{2(Q-\alpha)}{\gamma}-1},A_1^{\frac{2(Q-\alpha)}{\gamma}-1}\}].$$

If $\frac{2(Q-\alpha)}{\gamma}\leq 1$,

$$\begin{aligned}\mathbb{E}[(A_1+u^{-\nu}A_2)^{\frac{2(Q-\alpha)}{\gamma}}-A_1^{\frac{2(Q-\alpha)}{\gamma}}]&\leq u^{-\nu}\mathbb{E}[A_2A_1^{\frac{2(Q-\alpha)}{\gamma}-1}]\\&\overset{\text{Hölder}}{\leq} u^{-\nu}\mathbb{E}[A_2^p]^{1/p}\mathbb{E}[A_1^{\frac{p}{p-1}(\frac{2(Q-\alpha)}{\gamma}-1)}]^{(p-1)/p}<Cu^{-\nu},\end{aligned}$$

where $1<p<\frac{4}{\gamma^2}$ to ensure that $\mathbb{E}[A_2^p]$ is finite, and we know that $A_1\geq\frac{1}{2}\int_0^\infty e^{\frac{\gamma}{2}\mathcal{B}_s^{\frac{Q-\alpha}{2}}}e^{\frac{\gamma}{2}Y(e^{-s/2})}ds$ has negative moments. On the other hand, if $\frac{2(Q-\alpha)}{\gamma}>1$, then:

$$\mathbb{E}[(A_1+u^{-\nu}A_2)^{\frac{2(Q-\alpha)}{\gamma}}-A_1^{\frac{2(Q-\alpha)}{\gamma}}]\leq\frac{2(Q-\alpha)}{\gamma}u^{-\nu}\mathbb{E}[\rho(\frac{Q-\alpha}{2})^{\frac{2(Q-\alpha)}{\gamma}}]<Cu^{-\nu}.$$

The moment of $\rho(\frac{Q-\alpha}{2})$ is finite since $\frac{2(Q-\alpha)}{\gamma}<\frac{4}{\gamma^2}$. □

3.4.5 Special functions

We include here a detailed discussion on hypergeometric functions and on the special functions Γ_γ and G that we have used in our paper. First, let us discuss the theory of hypergeometric equations and the so-called connection formulas between the different bases of their solutions. For $A>0$ let $\Gamma(A)=\int_0^\infty t^{A-1}e^{-t}dt$ denote the standard Gamma function and let $(A)_n:=\frac{\Gamma(A+n)}{\Gamma(n)}$. For $A,B,C,$ and x real numbers we define the hypergeometric function F by:

$$F(A,B,C,x):=\sum_{n=0}^{\infty}\frac{(A)_n(B)_n}{n!(C)_n}x^n. \tag{3.4.49}$$

This function can be used to solve the following hypergeometric equation:

$$(t(1-t)\frac{d^2}{dt^2}+(C-(A+B+1)t)\frac{d}{dt}-AB)U(t)=0. \tag{3.4.50}$$

For our purposes we will always work with the parameter $t\in(-\infty,0)$ and we can give the following two bases of solutions,

$$\begin{aligned}U(t)&=C_1F(A,B,C,t)+C_2|t|^{1-C}F(1+A-C,1+B-C,2-C,t)\\&=D_1|t|^{-A}F(A,1+A-C,1+A-B,t^{-1})+D_2|t|^{-B}F(B,1+B-C,1+B-A,t^{-1}),\end{aligned}$$

where the first expression is an expansion in power of $|t|$ and the second is an expansion in powers of $|t|^{-1}$. For each basis we have two real constants that parametrize the solution space, C_1,C_2 and D_1,D_2. We thus expect to have an explicit change of basis formula that will give a link between C_1,C_2 and D_1,D_2. This is precisely what give the so-called connection formulas:

$$\begin{pmatrix}C_1\\C_2\end{pmatrix}=\begin{pmatrix}\frac{\Gamma(1-C)\Gamma(A-B+1)}{\Gamma(A-C+1)\Gamma(1-B)} & \frac{\Gamma(1-C)\Gamma(B-A+1)}{\Gamma(B-C+1)\Gamma(1-A)}\\ \frac{\Gamma(C-1)\Gamma(A-B+1)}{\Gamma(A)\Gamma(C-B)} & \frac{\Gamma(C-1)\Gamma(B-A+1)}{\Gamma(B)\Gamma(C-A)}\end{pmatrix}\begin{pmatrix}D_1\\D_2\end{pmatrix}. \tag{3.4.51}$$

This relation comes from the theory of hypergeometric equations and we will extensively use it to deduce our shift equations. Of course a similar formula holds for $\tilde{U}(t)$ and for the parameters $\tilde{A}, \tilde{B}, \tilde{C}$.

We will now provide some explanations on the function $\Gamma_\gamma(x)$ that we have introduced as well as its connection with the so-called G Barnes' function. Our conventions for the function $\Gamma_\gamma(x)$ follow the ones of the appendix of [57].[7] For all $\gamma \in (0,2)$ and for $x > 0$, $\Gamma_\gamma(x)$ is defined by the integral formula written in Theorem 3.1.1,

$$\ln \Gamma_\gamma(x) = \int_0^\infty \frac{dt}{t}\left[\frac{e^{-xt} - e^{-\frac{Qt}{2}}}{(1-e^{-\frac{\gamma t}{2}})(1-e^{-\frac{2t}{\gamma}})} - \frac{(\frac{Q}{2}-x)^2}{2}e^{-t} + \frac{x-\frac{Q}{2}}{t}\right], \tag{3.4.52}$$

where we have $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$. This function $\Gamma_\gamma(x)$ is completely determined by the following two shift equations,

$$\frac{\Gamma_\gamma(x)}{\Gamma_\gamma(x+\frac{\gamma}{2})} = \frac{1}{\sqrt{2\pi}}\Gamma(\frac{\gamma x}{2})(\frac{\gamma}{2})^{-\frac{\gamma x}{2}+\frac{1}{2}}, \tag{3.4.53}$$

$$\frac{\Gamma_\gamma(x)}{\Gamma_\gamma(x+\frac{2}{\gamma})} = \frac{1}{\sqrt{2\pi}}\Gamma(\frac{2x}{\gamma})(\frac{\gamma}{2})^{\frac{2x}{\gamma}-\frac{1}{2}}, \tag{3.4.54}$$

and by its value in $\frac{Q}{2}$, $\Gamma_\gamma(\frac{Q}{2}) = 1$. We mention that $\Gamma_\gamma(x)$ is an analytic function of x. In the case where $\gamma = 2$ the function $\Gamma_\gamma(x)$ reduces to,

$$\Gamma_2(x) = (2\pi)^{\frac{x}{2}-\frac{1}{2}}G(x)^{-1}, \tag{3.4.55}$$

where $G(x)$ is the so-called Barnes G function. This function is useful when we study the limit $\gamma \to 2$ in section 3.1.4. Finally in our Corollary 3.1.3 we have used a special $\beta_{2,2}$ distribution defined in [63]. Here we recall the definition:

Definition 3.4.12 (Existence theorem)**.** *The distribution* $-\ln \beta_{2,2}(a_1, a_2; b_0, b_1, b_2)$ *is infinitely divisible on* $[0,\infty)$ *and has the Lévy-Khintchine decomposition for* $\mathrm{Re}(p) > -b_0$*:*

$$\mathbb{E}[\exp(p \ln \beta_{2,2}(a_1,a_2;b_0,b_1,b_2))] = \exp\Big(\int_0^\infty (e^{-pt}-1)e^{-b_0 t}\frac{(1-e^{-b_1 t})(1-e^{-b_2 t})}{(1-e^{-a_1 t})(1-e^{-a_2 t})}\frac{dt}{t}\Big). \tag{3.4.56}$$

Furthermore, the distribution $\ln \beta_{2,2}(a_1, a_2; b_0, b_1, b_2)$ *is absolutely continuous with respect to the Lebesgue measure.*

We only work with the case $(a_1, a_2) = (1, \frac{4}{\gamma^2})$. Then the law $\beta_{2,2}(1, \frac{4}{\gamma^2}; b_0, b_1, b_2)$ depends on 4 parameters γ, b_0, b_1, b_2 and its real moments $p > -b_0$ are given by the formula:

$$\mathbb{E}[\beta_{2,2}(1,\frac{4}{\gamma^2};b_0,b_1,b_2)^p] = \frac{\Gamma_\gamma(\frac{\gamma}{2}(p+b_0))\Gamma_\gamma(\frac{\gamma}{2}(b_0+b_1))\Gamma_\gamma(\frac{\gamma}{2}(b_0+b_2))\Gamma_\gamma(\frac{\gamma}{2}(p+b_0+b_1+b_2))}{\Gamma_\gamma(\frac{\gamma}{2}b_0)\Gamma_\gamma(\frac{\gamma}{2}(p+b_0+b_1))\Gamma_\gamma(\frac{\gamma}{2}(p+b_0+b_2))\Gamma_\gamma(\frac{\gamma}{2}(b_0+b_1+b_2))}. \tag{3.4.57}$$

Of course we have $\gamma \in (0,2)$ and the real numbers p, b_0, b_1, b_2 must be chosen so that the arguments of all the Γ_γ are positive. We conclude this section with a few computations that we need that also involve hypergeometric functions.

[7] In [63] Ostrovsky uses a slightly different special function $\Gamma_2(x|\tau)$, the relation with our $\Gamma_\gamma(x)$ is:

$$\Gamma_\gamma(x) = (\frac{2}{\gamma})^{\frac{1}{2}(x-\frac{Q}{2})^2}\frac{\Gamma_2(\frac{2x}{\gamma}|\tau)}{\Gamma_2(\frac{Q}{\gamma}|\tau)}.$$

Lemma 3.4.13. *For $p < 0$ and $-1 < a < 0$ or for $0 < p < 1$ and $-1 < a < -p$ we have the identity:*

$$\int_0^\infty ((u+1)^p - 1)u^{a-1}du = \frac{\Gamma(a)\Gamma(-a-p)}{\Gamma(-p)}. \tag{3.4.58}$$

Proof. Denote by $(x)_n := x(x+1)\dots(x+n-1)$.

$$\begin{aligned}\int_0^\infty ((u+1)^p - 1)u^{a-1}du &= \sum_{n=0}^\infty \frac{(-1)^n}{n!}(-p)_n \frac{1}{n+a} - \sum_{n=0}^\infty \frac{(-1)^n}{n!}(-p)_n \frac{1}{a+p-n} \\ &= \frac{1}{a}\sum_{n=0}^\infty \frac{(-1)^n}{n!}\frac{(-p)_n(a)_n}{(a+1)_n} - \frac{1}{a+p}\sum_{n=0}^\infty \frac{(-1)^n}{n!}\frac{(-p)_n(-a-p)_n}{(-a-p+1)_n} \\ &= \frac{1}{a}F(-p,a,a+1,-1) - \frac{1}{a+p}F(-p,-a-p,-a-p+1,-1) \\ &= \frac{\Gamma(a)\Gamma(-a-p)}{\Gamma(-p)},\end{aligned}$$

where in the last line we used the formula, for suitable $\overline{a}, \overline{b} \in \mathbb{R}$,

$$\bar{b}F(\bar{a}+\bar{b},\bar{a},\bar{a}+1,-1) + \bar{a}F(\bar{a}+\bar{b},\bar{b},\bar{b}+1,-1) = \frac{\Gamma(\bar{a}+1)\Gamma(\bar{b}+1)}{\Gamma(\bar{a}+\bar{b})}.$$

□

Lemma 3.4.14. *For $0 < a < 1 - \frac{\gamma^2}{4}$ we have:*

$$\frac{\gamma^2}{4}\int_0^\infty (y+1)^{\frac{\gamma^2}{4}-1}y^{a-1}dy = (a+\frac{\gamma^2}{4})\frac{\Gamma(a)\Gamma(-a-\frac{\gamma^2}{4})}{\Gamma(-\frac{\gamma^2}{4})}. \tag{3.4.59}$$

Proof. By the previous lemma,

$$\int_0^\infty ((y+z)^{\frac{\gamma^2}{4}} - 1)y^{a-1}dy = z^{a+\frac{\gamma^2}{4}}\frac{\Gamma(a)\Gamma(-a-\frac{\gamma^2}{4})}{\Gamma(-\frac{\gamma^2}{4})}.$$

We take the derivative in z in the above equation and evaluate it at $z = 1$ to get:

$$\frac{\gamma^2}{4}\int_0^\infty (y+1)^{\frac{\gamma^2}{4}-1}y^{a-1}dy = (a+\frac{\gamma^2}{4})\frac{\Gamma(a)\Gamma(-a-\frac{\gamma^2}{4})}{\Gamma(-\frac{\gamma^2}{4})}.$$

□

Chapter 4

Liouville quantum gravity on the annulus

In this work we construct Liouville quantum gravity on an annulus in the complex plane. This construction is aimed at providing a rigorous mathematical framework to the work of theoretical physicists initiated by Polyakov in 1981 [66]. It is also a very important example of a conformal field theory (CFT). Results have already been obtained on the Riemann sphere [17] and on the unit disk [40] so this paper will follow the same approach. The case of the annulus contains two difficulties: it is a surface with two boundaries and it has a non trivial moduli space. We recover the Weyl anomaly - a formula verified by all CFT - and deduce from it the KPZ formula. We also show that the full partition function of Liouville quantum gravity integrated over the moduli space is finite. This allows us to give the joint law of the Liouville measures and of the random modulus and to write the conjectured link with random planar maps.

4.1 Introduction

The goal of this work is to provide a rigorous probabilistic construction of the theory of Liouville quantum gravity (LQG) on a surface with the topology of an annulus. This theory was first introduced by the physicist A. Polyakov in 1981 in his seminal paper "Quantum Geometry of Bosonic Strings", see [66]. The case of surfaces with boundary was studied in [2] and the precise case of the annulus was studied in [52]. The probabilistic framework used throughout this paper was introduced in [17] where the authors provide a construction of LQG for the Riemann sphere. Following the same approach, the theory has been defined on the unit disk in [40], on the complex tori in [18] and on compact Riemann surfaces of higher genus in [37]. The Riemann sphere is the simplest case as it corresponds to a simply connected compact surface without boundary. When considering the unit disk, technical difficulties appear due to the presence of the boundary and extra boundary terms have to be added in the Liouville action. The torus and higher genus surfaces have a non-trivial moduli space, meaning that all tori are not equivalent under conformal maps. In this case defining LQG requires to integrate over the moduli space with an appropriate measure. The annulus possesses two boundaries and has a non-trivial moduli space so we will encounter both of these difficulties.

The first building block of Liouville quantum gravity is the Liouville quantum field theory (LQFT), which in probabilistic terms corresponds to giving the law of a random field ϕ on an annulus Ω of radii 1 and τ.[1] To make the construction clearer, let us make an analogy with Brownian motion. Physicists often define the law of Brownian motion using the Feynman path integral representation. Informally, let Σ_1 be the space of paths $\sigma : [0,1] \to \mathbb{R}$ that start from $\sigma(0) = 0$. We can define the following functional on Σ_1 by:

$$\forall \sigma \in \Sigma_1,\ S_{BM}(\sigma) = \frac{1}{2}\int_0^1 |\dot{\sigma}(t)|^2 dt. \tag{4.1.1}$$

This functional is minimal for $\sigma \equiv 0$, this is the "classical" solution. The quantum theory corresponds to considering all paths of Σ_1 with a probability density given by the exponential of $-S_{BM}$: S_{BM} is the energy of the path, $e^{-S_{BM}}$ is the corresponding Boltzmann weight. This leads to the formal path integral definition of Brownian motion, for all suitable functionals F,

$$\mathbb{E}[F((B_s)_{0\le s\le 1})] = \frac{1}{\mathcal{Z}}\int_{\Sigma_1} F(\sigma)e^{-S_{BM}(\sigma)}D\sigma, \tag{4.1.2}$$

where $D\sigma$ represents a formal uniform measure on Σ_1 and $\mathcal{Z}$ is a normalization constant. Although $D\sigma$ and $S_{BM}(\sigma)$ are ill-defined (a typical path σ will not be differentiable), it is possible to give a rigorous meaning to this expression by subdividing $[0,1]$ into n points and taking n to infinity. The fact that we recover the law of Brownian motion is due to the Donsker theorem.

Brownian motion is often seen as the canonical uniform random path in $\mathbb{R}^d$: it is the scaling limit of the simple random walk on the regular Euclidean lattice and it appears in a wide variety of problems in probability. A natural question is to ask what would be the equivalent in two dimensions, what is the canonical random geometry of a surface of given topology? More precisely, what is the canonical random Riemannian metric on a given surface? The answer is given by the Liouville quantum field theory. Just like for Brownian motion, there are two possible approaches to LQFT: one continuum approach that uses the Feynman integral formalism and one that uses the scaling limit of discrete models called random planar maps. We will focus on the first approach and construct directly the continuous object. For a given background Riemannian metric g on our annulus Ω, we want to consider the formal random metric $e^{\gamma\phi}g$ where γ is a positive real number and where ϕ is the Liouville field whose law is formally given for all suitable functionals F by,

$$\mathbb{E}[F(\phi)] = \frac{1}{\mathcal{Z}}\int_{\Sigma} F(X)e^{-S_L(X,g)}D_gX, \tag{4.1.3}$$

with D_gX being a formal uniform measure on the space Σ of maps $X : \Omega \to \mathbb{R}$. $S_L(X,g)$ is the Liouville action defined for each map X of Σ and each metric g on Ω by,

$$S_L(X,g) = \frac{1}{4\pi}\int_{\Omega}(|\partial^g X|^2 + QR_gX + 4\pi\mu e^{\gamma X})d\lambda_g + \frac{1}{2\pi}\int_{\partial\Omega}(QK_gX + 2\pi\mu_\partial e^{\frac{\gamma}{2}X})d\lambda_{\partial g}, \tag{4.1.4}$$

where ∂_g, R_g, K_g, $d\lambda_g$, and $d\lambda_{\partial g}$ stand respectively for the gradient, Ricci scalar curvature, geodesic curvature (along the boundary), volume form and line element along $\partial\Omega$ in the metric g. The parameters μ, $\mu_\partial \ge 0$ (with $\mu + \mu_\partial > 0$) are respectively the bulk and boundary cosmological constants and Q, γ satisfy $\gamma \in (0,2)$ and $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$. We notice that there are two boundary terms in our action which would not be present if Ω had no boundary.

[1] $\tau \in (1, +\infty)$ parametrizes the space of non-conformally equivalent annuli. To define LQFT we work at fixed τ and then to define LQG we will perform an integration over τ.

One may wonder why the Liouville action is the correct action to define canonical random metrics. A first answer comes from physics, in particular from Polyakov in [66] (see the appendix for some heuristics). An easier answer comes from the study of classical Liouville theory, meaning that we look for the functions X minimizing the Liouville action. It is a well known fact of classical geometry that such a minimum X_{min} is unique if it exists and the new metric $g' = e^{\gamma X_{min}} g$ on Ω is of constant negative curvature provided that $Q_c = \frac{2}{\gamma}$[2]. In other words, the minimum of the Liouville action uniformizes the surface (Ω, g) and it is therefore natural to look at quantum fluctuations of the uniformized metric $e^{\gamma X_{min}} g$. This is precisely the meaning of (4.1.3).

As we have written it the path integral (4.1.3) diverges for any surface of genus 0 or 1, including therefore the case of the annulus. To see this we can write the Gauss-Bonnet formula, given here for a boundaryless surface M of genus h: $\int_M R_g d\lambda_g = 8\pi(1-h)$. When $h = 0$ or 1, this implies that it is impossible to define on the surface a metric of constant negative curvature, meaning that $S_L(X, g)$ will have no minimum and therefore the path integral (4.1.3) diverges. To solve this problem we proceed as in [40] and add insertion points. We consider the new expression,

$$\mathbb{E}[F(\phi)] = \frac{1}{\mathcal{Z}} \int_\Sigma F(X) e^{\sum_{i=1}^n \alpha_i X(z_i) + \frac{1}{2}\sum_{j=1}^{n'} \beta_j X(s_j)} e^{-S_L(X,g)} D_g X, \tag{4.1.5}$$

where we have chosen n insertion points $z_i \in \Omega$ with weights $\alpha_i \in \mathbb{R}$ in the interior of the annulus and n' insertions points $s_j \in \partial\Omega$ with weights $\beta_j \in \mathbb{R}$ on the boundaries of the annulus ($n, n' \in \mathbb{N}$). We show that the following conditions known as the Seiberg bounds must be satisfied in order for (4.1.5) to exist:

$$\sum_{i=1}^n \alpha_i + \sum_{j=1}^{n'} \frac{\beta_j}{2} > 0 \quad \text{and} \quad \forall i,\ \alpha_i < Q, \quad \forall j,\ \beta_j < Q. \tag{4.1.6}$$

Theses bounds imply that the minimum number of insertion points is one point in Ω or one point on the boundary $\partial\Omega$. Fixing a point inside Ω is actually a stronger constraint than fixing a boundary point. Also note that choosing one boundary point is precisely the requirement to entirely determine a conformal automorphism of the annulus. From a geometric standpoint, we can also view insertion points as conical singularities of the metric which allow hyperbolic metrics to be defined on the surface.

Let us now outline the structure of our paper. We start in section 4.2 by introducing the Gaussian free field and the associated Gaussian multiplicative chaos measures and state the properties of these objects that we will need in the sequel. Then in section 4.3 we give a mathematical definition for the partition function of LQFT formally given by (4.1.5). We establish some of its properties well known to physicists such as the Weyl anomaly (behavior under change of metric) and the KPZ formula (behavior under conformal automorphism). In the subsection 4.3.4 we define the Liouville field and the Liouville bulk and boundary measures for a fixed value of τ. Lastly in section 4.4 we construct the full theory of LQG by integrating over $\tau \in (1, +\infty)$ the partition function of LQFT along with the partition function of a matter field.[3] We prove that this integral over moduli space is convergent and thus we show that the partition function of LQG is well defined. We then state

[2]Here we write $Q_c = \frac{2}{\gamma}$ as this is the correct value in the classical theory where the goal is to minimize the Liouville action. In the quantum theory we will always have $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$.

[3]All partition function will first be computed on an annulus of radii 1 and τ and then we will perform the integration over τ.

the joint law of the Liouville measures and of the random modulus and write the conjectured link with the scaling limit of random planar maps.

Our paper also clarifies a few points that were not addressed in [17, 40]. First, we justify for any background metric g the expression of the covariance of our field X_g by diagonalizing an operator T defined using Neumann boundary conditions. We also reconcile two possible approaches to Liouville conformal field theory - the one proposed in [17, 40] and the framework of Gawedzki [36] - by explaining that these two approaches correspond to different ways of introducing the metric dependence of the theory. This allows to provided a very simple proof of the KPZ formula which becomes a direct consequence of the Weyl anomaly.

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4.2 Gaussian free field and Gaussian multiplicative chaos

4.2.1 Geometric background

The first tool we need is a description of the space of metrics on a surface with the topology of an annulus. The uniformization theorem tells us that any metric g can be written up to a change of coordinates $e^{\varphi}dx^2$ where e^{φ} is the Weyl factor and dx^2 is the Euclidean metric on an annulus $\Omega = \{z, 1 < |z| < \tau\}$ with radii 1 and τ, the value of τ being uniquely determined. The parameter $\tau \in (1, +\infty)$ parametrizes the space of non conformally equivalent annuli known as the moduli space. We will always work at fixed τ with the exception of in section 4.4 where will perform an integration over τ.

For the annulus given by $\Omega = \{z, 1 < |z| < \tau\}$, we consider the inner and outer boundaries $\partial\Omega_1 = \{z, |z| = 1\}$ and $\partial\Omega_\tau = \{z, |z| = \tau\}$ and we set $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_\tau$. When Ω and $\partial\Omega$ are equipped with the Euclidean metric, we will write $d\lambda$ and $d\lambda_\partial$ for the Lebesgue measures on Ω and $\partial\Omega$, Δ for the Laplace-Beltrami operator and ∂_n for the Neumann operator. The convention for ∂_n is that the derivative is computed along the vector $\overrightarrow{n}$ normal to the boundary and pointing outward of Ω. For a general metric g we will use the notations ∂^g, Δ_g, ∂_{n_g}, R_g, K_g, $d\lambda_g$, and $d\lambda_{\partial g}$ for the gradient, Laplace-Beltrami operator, Neumann operator, Ricci scalar curvature, geodesic curvature, volume form and line element along $\partial\Omega$ in the metric g. Under a Weyl rescaling of the metric $g' = e^{\varphi}g$, these objects behave in the following way: $\Delta_{g'} = e^{-\varphi}\Delta_g$, $\partial_{n_{g'}} = e^{-\varphi/2}\partial_{n_g}$, $d\lambda_{g'} = e^{\varphi}d\lambda_g$, and $d\lambda_{\partial g'} = e^{\varphi/2}d\lambda_{\partial g}$. We also have:

$$R_{g'} = e^{-\varphi}(R_g - \Delta_g\varphi) \tag{4.2.1}$$

$$K_{g'} = e^{-\varphi/2}(K_g + \partial_{n_g}\varphi/2) \tag{4.2.2}$$

These formulas will be useful for $g = e^{\varphi}dx^2$. In the case of the Euclidean metric dx^2 we have $R = 0$ on Ω, $K = -1$ on $\partial\Omega_1$, and $K = \frac{1}{\tau}$ on $\partial\Omega_\tau$. Let us recall the Gauss-Bonnet theorem,

$$\int_\Omega R_g d\lambda_g + 2\int_{\partial\Omega} K_g d\lambda_{\partial g} = 4\pi\chi(\Omega) = 0, \tag{4.2.3}$$

where in the case of the annulus the Euler characteristics $\chi(\Omega)$ is worth 0. Similarly, the Green-Riemann formula gives:

$$\int_\Omega \psi\, \Delta_g\varphi\, d\lambda_g + \int_\Omega \partial^g\varphi \cdot \partial^g\psi\, d\lambda_g = \int_{\partial\Omega} \partial_{n_g}\varphi\, \psi\, d\lambda_{\partial g}. \tag{4.2.4}$$

Here again the derivative ∂_{n_g} is along the normal vector pointing outwards of Ω. For a function f defined on Ω or $\partial\Omega$, we define the averages $m_g(f)$ and $m_{\partial g}(f)$ with respect to $d\lambda_g$ and $d\lambda_{\partial g}$ as $m_g(f) = \frac{1}{\lambda_g(\Omega)}\int_\Omega f d\lambda_g$ and $m_{\partial\Omega}(f) = \frac{1}{\lambda_{\partial g}(\partial\Omega)}\int_{\partial\Omega} f d\lambda_{\partial g}$. $m(f)$ and $m_\partial(f)$ will correspond to the averages in the flat metric. Finally let us mention that the conformal automorphisms of our annulus Ω are the rotations plus the inversion that exchanges both boundaries. Therefore a conformal automorphism can be written either $z \to e^{i\theta}z$ or $z \to e^{i\theta}\frac{\tau}{z}$ for a $\theta \in [0, 2\pi)$.

4.2.2 Gaussian free field

The first step in the construction of LQFT is to interpret the formal density $D_g X e^{-\frac{1}{4\pi}\int_\Omega |\partial^g X|^2 d\lambda_g}$ as the density of a Gaussian free field (GFF) with certain boundary conditions. Let $H_0^1(\Omega)$ be the Sobolev space on Ω with $\forall f \in H_0^1(\Omega)$, $\int_{\partial\Omega} f d\lambda_{\partial g} = 0$. $H_0^1(\Omega)$ is a Hilbert space with respect to the inner product given for $f, h \in H_0^1(\Omega)$ by:

$$\langle f, h\rangle := \int_\Omega (\partial f \cdot \partial h) d\lambda. \tag{4.2.5}$$

We define the following operator T on $H_0^1(\Omega)$ by, $\forall f \in H_0^1(\Omega), T(f) = u$ where the function u is the unique solution to the following Neumann problem:

$$\begin{cases} \Delta_g u(z) = -2\pi f(z) & \text{for} \quad z \in \Omega \\ \partial_{n_g} u(z) = -\frac{c_g(z)}{\tau+1}\int_\Omega f(z') d\lambda_g(z') & \text{for} \quad z \in \partial\Omega \\ \int_{\partial\Omega} u(z) d\lambda_{\partial g}(z) = 0 \end{cases} \tag{4.2.6}$$

The function $c_g(z)$ defined on $\partial\Omega$ that appears in the above is equal to 1 in the case of the Euclidean metric and has a general expression in any metric g given by (4.2.24). We can give the expression of u in terms of f using the Green's function G_g:

$$u(z) = \int_\Omega G_g(z, z') f(z') d\lambda_g(z'). \tag{4.2.7}$$

The expression for G_g will be given in subsection 4.2.3. T is an auto-adjoint compact operator of $H_0^1(\Omega)$, therefore we can find a basis $e_i, i \in \mathbb{N}^*$ of eigenvectors with associated eigenvalue $\lambda_i^{-1} > 0$. The e_i are normalized so that $\int_\Omega e_i(z)^2 d\lambda_g(z) = 1$. With this basis of eigenfunctions we can formally write,

$$D_g X e^{-\frac{1}{4\pi}\int_\Omega |\partial^g X|^2 d\lambda_g} = \prod_{i=1}^{+\infty} dx_i \, e^{-\frac{1}{2}\sum_{i=1}^{+\infty}\lambda_i x_i^2}, \tag{4.2.8}$$

where each dx_i is a Lebesgue measure on $\mathbb{R}$. This corresponds to the density of a GFF X_g which can be written,[4]

$$X_g(z) = \sum_{i=1}^{+\infty} \frac{x_i e_i(z)}{\sqrt{\lambda_i}}, \tag{4.2.9}$$

where in this expression the x_i are i.i.d. standard Gaussian variables. This sum converges in the Sobolev space $H^{-1}(\Omega)$. Just like for all the e_i, we have $\int_{\partial\Omega} X_g \, d\lambda_{\partial g} = 0$. But this condition is arbitrary as the formal density $D_g X e^{-\frac{1}{4\pi}\int_\Omega |\partial^g X|^2 d\lambda_g}$ only defines X_g up to a constant. To deal with this problem we will actually consider $X_g + c$, where X_g is the GFF defined above with zero

[4] Here the dependence in g is contained in e_i and in λ_i.

mean on the boundary and c is a constant integrated according to the Lebesgue measure on $\mathbb{R}$. We must also discuss the value of the formal normalization constant $\mathcal{Z}_{GFF}(g)$ of our Gaussian density,

$$\mathcal{Z}_{GFF}(g) = \int_{\Sigma'} D_g X e^{-\frac{1}{4\pi}\int_\Omega |\partial^g X|^2 d\lambda_g}, \tag{4.2.10}$$

where here Σ' stands for the space of maps $X : \Omega \mapsto \mathbb{R}$ with zero average on $\partial\Omega$. The dependence of $\mathcal{Z}_{GFF}(g)$ on the background metric g is given by (4.3.2). To summarize this construction we have for suitable functionals $\hat{F}$:

$$\int_\Sigma D_g X e^{-\frac{1}{4\pi}\int |\partial^g X|^2 d\lambda_g} \hat{F}(X) = \mathcal{Z}_{GFF}(g) \int_\mathbb{R} dc\, \mathbb{E}[\hat{F}(X_g + c)]. \tag{4.2.11}$$

To define LQFT we will choose,

$$\hat{F}(X) = F(X) e^{\sum_i \alpha_i X(z_i) + \frac{1}{2}\sum_j \beta_j X(s_j)} e^{-\frac{1}{4\pi}\int_\Omega (Q R_g X + 4\pi\mu e^{\gamma X}) d\lambda_g} e^{-\frac{1}{2\pi}\int_{\partial\Omega}(Q K_g X + 2\pi\mu_\partial e^{\frac{\gamma}{2}X}) d\lambda_{\partial g}}, \tag{4.2.12}$$

where F is again some functional, the (z_i, α_i) are the bulk insertion points, the (s_j, β_j) are the boundary insertion points, and the last part corresponds to the curvature and exponential terms of the Liouville action. Extra care will be required as X_g lives in the space of distributions $H^{-1}(\Omega)$ so $e^{\gamma X_g}$ is ill-defined.

4.2.3 Properties of the Green's function

We can check that the function $G_g(z, z')$, which solves problem (4.2.6) through the relation (4.2.7), is also the covariance of our Gaussian free field defined by (4.2.9):

$$G_g(z, z') = \mathbb{E}[X_g(z) X_g(z')] = \sum_{i=1}^\infty \frac{e_i(z) e_i(z')}{\lambda_i}. \tag{4.2.13}$$

The Neumann boundary conditions of (4.2.6) entirely determine G_g. Other boundary conditions would also be possible but they lead to major inconsistencies in the theory[5]. In the case of the Euclidean metric $g = dx^2$, we simply write G for the Green's function. From the computation detailed in the appendix we obtain the following expression for G using polar coordinates $z = re^{i\theta}$, $z' = \rho e^{i\phi}$,

$$G(r, \theta, \rho, \phi) = g_0(r, \rho) + 2\sum_{n=1}^\infty g_n(r, \rho) \cos n(\theta - \phi) + \ln \frac{|\tau^4 z^2 z'^2|}{|1 - z\overline{z'}||\tau^2 - z\overline{z'}||z - z'||\tau^2 z - z'|}, \tag{4.2.14}$$

where the factor $|\tau^2 z - z'|$ holds for $r < \rho$ and is replaced by $|z - \tau^2 z'|$ for $r > \rho$, and where

$$g_n(r, \rho) = \frac{r^{-n}\rho^{-n}}{2n\tau^{2n}(\tau^{2n} - 1)} \begin{cases} (\tau^{2n} + \rho^{2n})(r^{2n} + 1), & \text{for} \quad r \le \rho \\ (\tau^{2n} + r^{2n})(\rho^{2n} + 1), & \text{for} \quad r \ge \rho \end{cases} \tag{4.2.15}$$

$$g_0(r, \rho) = \begin{cases} \frac{\ln(r) + \tau^2 \ln(\tau/\rho) + \tau \ln(r/\rho)}{(\tau+1)^2}, & \text{for} \quad r \le \rho \\ \frac{\ln(\rho) + \tau^2 \ln(\tau/r) + \tau \ln(\rho/r)}{(\tau+1)^2}, & \text{for} \quad r \ge \rho \end{cases} \tag{4.2.16}$$

[5] For instance for Dirichlet boundary conditions the Weyl anomaly of section 4.3.2 fails to hold.

We now give some useful properties of the Green's function. By construction we have:

$$G(z,z') = G(z',z) \tag{4.2.17}$$
$$\Delta G(z,z') = -2\pi\delta_0(z-z') \tag{4.2.18}$$

By direct computation on the expression of the Green's function, we have the following results:

$$\begin{aligned}\frac{\partial G(r,\theta,\rho,\phi)}{\partial r}|_{r=1} &= \frac{\partial g_0(r,\rho)}{\partial r}|_{r=1} = \begin{cases}\frac{1}{\tau+1}, & \text{for} \quad \rho>1\\ \frac{-\tau}{\tau+1}, & \text{for} \quad \rho=1\end{cases}\\ \frac{\partial G(r,\theta,\rho,\phi)}{\partial r}|_{r=\tau} &= \frac{\partial g_0(r,\rho)}{\partial r}|_{r=\tau} = \begin{cases}\frac{-1}{\tau+1}, & \text{for} \quad \rho<\tau\\ \frac{1}{\tau(\tau+1)}, & \text{for} \quad \rho=\tau\end{cases}\end{aligned} \tag{4.2.19}$$

Concerning the integral of the Green's function over the boundaries, we get:

$$\int_{\partial\Omega} G(z,z')d\lambda_\partial(z) = \frac{2\pi}{(\tau+1)^2}(\tau^2\ln(\tau/\rho)+\tau\ln(1/\rho)) + \frac{2\pi\tau}{(\tau+1)^2}(\ln(\rho)+\tau\ln(\rho/\tau)) = 0. \tag{4.2.20}$$

Finally, we have the following formula valid for all (smooth) functions f on Ω and for all $z\in\Omega$:

$$\int_\Omega G(z,z')\Delta f(z')d\lambda(z') - \int_{\partial\Omega} G(z,z')\partial_n f(z')d\lambda_\partial(z') = -2\pi(f(z)-m_\partial(f)). \tag{4.2.21}$$

Indeed, using (4.2.4) and (4.2.19)

$$\int_\Omega G(z,z')\Delta f(z')d\lambda(z') = -\int_\Omega \partial G(z,z')\partial f(z')d\lambda(z') + \int_{\partial\Omega} G(z,z')\partial_n f(z')d\lambda_\partial(z')$$

and

$$\begin{aligned}-\int_\Omega \partial G(z,z')\partial f(z')d\lambda(z') &= \int_\Omega \Delta G(z,z')f(z')d\lambda(z') - \int_{\partial\Omega}\partial_n G(z,z')f(z')d\lambda_\partial(z')\\ &= -2\pi f(z) + \frac{1}{\tau+1}\int_{\partial\Omega} f(z')d\lambda_\partial(z')\\ &= -2\pi(f(z)-m_\partial(f)).\end{aligned}$$

We now discuss the relation between G_g and G when $g=e^\varphi dx^2$. By definition G solves our Neumann problem (4.2.6) in the flat metric dx^2. We will show that G_g is given by the following expression,

$$G_g(z,z') = G(z,z') - m_{\partial g}(G(z,\cdot)) - m_{\partial g}(G(\cdot,z')) + m_{\partial g}(G(\cdot,\cdot)),^6 \tag{4.2.22}$$

which means that we must check that this function solves our problem (4.2.6) in the metric g. By construction of G_g, we clearly have $\int_{\partial\Omega} u(z)d\lambda_{\partial g}(z) = 0$. Next we must check that $\Delta_g u(z) = -2\pi f(z)$:

$$\begin{aligned}\Delta_g u(z) &= e^{-\varphi(z)}\int_\Omega \Delta G(z,z')f(z')d\lambda_g(z') - e^{-\varphi(z)}\int_\Omega \Delta(m_{\partial g}G(z,\cdot))f(z')d\lambda_g(z')\\ &= -2\pi f(z) - e^{-\varphi(z)}\frac{\lambda_g(\Omega)}{\lambda_{\partial g}(\partial\Omega)}m_g(f)\int_{\partial\Omega}\Delta G(z,z')d\lambda_{\partial g}(z')\\ &= -2\pi f(z).\end{aligned}$$

[6] Here we have defined $m_{\partial g}(G(\cdot,\cdot)) := \frac{1}{\lambda_{\partial g}(\partial\Omega)^2}\int_{\partial\Omega}\int_{\partial\Omega} G(z,z')d\lambda_{\partial g}(z)d\lambda_{\partial g}(z')$.

Lastly we look at the normal derivative, the function $c_g(z)$ introduced in (4.2.6) will be chosen to match the following computation,

$$\begin{aligned}\partial_{n_g} u(z) &= e^{-\varphi(z)/2}\int_\Omega \partial_n G(z,z')f(z')d\lambda_g(z') - e^{-\varphi(z)/2}\int_\Omega \partial_n m_{\partial g} G(z,\cdot)f(z')d\lambda_g(z') \\ &= -\frac{e^{-\varphi(z)/2}}{\tau+1}\int_\Omega f(z')d\lambda_g(z') - e^{-\varphi(z)/2}\partial_n m_{\partial g}G(z,\cdot)\int_\Omega f(z')d\lambda_g(z') \\ &= -\frac{c_g(z)}{\tau+1}\int_\Omega f(z')d\lambda_g(z'), \end{aligned} \tag{4.2.23}$$

where a straightforward computation shows that $c_g(z)$ is given by:

$$c_g(z) = \begin{cases} e^{-\varphi(z)/2}(1+\tau\lambda_{\partial g}(\partial\Omega_1) - \lambda_{\partial g}(\partial\Omega_\tau)), & \text{for} \quad z\in\partial\Omega_1 \\ e^{-\varphi(z)/2}(1-\lambda_{\partial g}(\partial\Omega_1) + \frac{1}{\tau}\lambda_{\partial g}(\partial\Omega_\tau)), & \text{for} \quad z\in\partial\Omega_\tau \end{cases} \tag{4.2.24}$$

We notice that $c_g(z) = 1$ for the Euclidean metric. From all the above we have shown that (4.2.22) gives the correct expression for G_g. From this we can easily deduce

$$X_g(z) \overset{law}{=} X(z) - m_{\partial g}(X) \tag{4.2.25}$$

where we have written X for the GFF in the flat metric of covariance G. In particular this tells us that $X_g + c$ for $g = e^\varphi dx^2$ and c distributed according to the Lebesgue measure on $\mathbb{R}$ is independent of φ because we can simply make a shift in the integral over c to remove the constant $m_{\partial g}(X)$. More precisely, for $g = e^\varphi dx^2$ and any suitable functional F:

$$\int_\mathbb{R} dc\, \mathbb{E}[F(X_g + c)] = \int_\mathbb{R} dc\, \mathbb{E}[F(X+c)]. \tag{4.2.26}$$

4.2.4 Circle average regularization

Let X be the GFF on Ω with covariance function given by G. Because X lives almost surely in the space of distributions $H^{-1}(\Omega)$ and we wish to define the exponential of X, we need to introduce a regularization procedure. We will choose a circle average regularization, more precisely for $\epsilon > 0$ we call $l_\epsilon(x)$ the length of the arc $A_\epsilon(x) = \{z\in\Omega; |z-x| = \epsilon\}$ and we set:

$$X_{g,\epsilon}(x) = \frac{1}{l_\epsilon(x)}\int_{A_\epsilon(x)} X(x + g(x)^{-1/2}s)ds. \tag{4.2.27}$$

For a point x at a distance larger than ϵ from the boundary this definition gives the standard circle average. The term $g^{-1/2}$ is added because the regularization must depend on the background metric g in order to get a consistent theory.[7] Here $g(x) = e^{\varphi(x)}$ with as usual $g = e^\varphi dx^2$. We have the following result:

Proposition 4.2.1. *As $\epsilon \to 0$ we have the following convergences.*
1) Uniformly over all compact subsets of Ω:

$$\mathbb{E}[X_{g,\epsilon}(x)^2] + \ln\epsilon \underset{\epsilon\to 0}{\to} \frac{1}{2}\ln g(x) + \ln g_P(x) + h(x)$$

[7] This way of introducing the metric dependence is slightly different than the one of [17, 40]. The advantage of our method is that it allows us to recover exactly the framework of [36], see Theorem 4.3.2. The KPZ formula of section 4.3.3 then becomes a direct consequence of the Weyl anomaly of section 4.3.2.

2) Uniformly over $\partial\Omega$:

$$\mathbb{E}[X_{g,\epsilon}(x)^2] + 2\ln\epsilon \underset{\epsilon\to 0}{\to} \ln g(x) + h_\partial(x)$$

Here g is our background metric, $g_P(x) := \frac{1}{|1-|x|^2||\tau^2-|x|^2|}$ is the term that diverges on the boundaries, and h, h_∂ are continuous functions independent of g whose explicit expressions are given in the proof below.

Proof. This proposition follows from an explicit computation with the Green's function of X. For a point $x \in \Omega$, we have:

$$\begin{aligned}\mathbb{E}[X_{g,\epsilon}(x)^2] &= \mathbb{E}[\frac{1}{4\pi^2}\int_0^{2\pi} X(x+\epsilon g(x)^{-1/2}e^{i\theta})d\theta \int_0^{2\pi} X(x+\epsilon g(x)^{-1/2}e^{i\theta'})d\theta'] \\ &= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} G(x+\epsilon g(x)^{-1/2}e^{i\theta}, x+\epsilon g(x)^{-1/2}e^{i\theta'})d\theta d\theta'\end{aligned}$$

We then notice that in the expression of the Green's function, only the term $\ln\frac{1}{|z-z'|}$ diverges when ϵ goes to 0. We have added consequently the term $\ln\epsilon$ to cancel this divergence. From this computation we get the expression of the function h,

$$\begin{aligned}h(x) &= g_0(r,r) + 2\sum_{n=1}^{\infty} g_n(r,r) + \ln\frac{\tau^4 r^3}{|\tau^2-1|} + \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} d\theta d\theta' \ln\frac{1}{|e^{i\theta}-e^{i\theta'}|} \\ &= g_0(r,r) + 2\sum_{n=1}^{\infty} g_n(r,r) + \ln\frac{\tau^4 r^3}{|\tau^2-1|},\end{aligned}$$

where $r = |x|$. Similarly for the boundary $\partial\Omega$ we have the following expression for h_∂:

$$h_\partial(x) = \begin{cases} g_0(1,1) + 2\sum_{n=1}^{\infty} g_n(1,1) + \ln\frac{|\tau^4|}{|\tau^2-1|^2} - \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} \ln|(e^{i\theta}-e^{i\theta'})\frac{\overline{x}e^{i\theta}+xe^{-i\theta'}}{x}|d\theta d\theta', & \text{for} \quad x \in \partial\Omega_1 \\ g_0(\tau,\tau) + 2\sum_{n=1}^{\infty} g_n(\tau,\tau) + \ln\frac{|\tau^6|}{|\tau^2-1|^2} - \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} \ln|(e^{i\theta}-e^{i\theta'})\frac{\overline{x}e^{i\theta}+xe^{-i\theta'}}{x}|d\theta d\theta', & \text{for} \quad x \in \partial\Omega_\tau \end{cases}$$

□

4.2.5 Gaussian multiplicative chaos

Now that our field X is random distribution, we need to tackle the problem of giving sense to the terms $\mu e^{\gamma X}$ and $\mu_\partial e^{\frac{\gamma}{2}X}$ as the exponential of a distribution is ill-defined. This can be done using the theory Gaussian multiplicative chaos which was introduced by Kahane in [42] and developed by others, see [72, 73]. The results from these papers allow us to claim:

Proposition 4.2.2. *Let $\gamma \in (0,2)$ and let g be a metric on Ω. We introduce for $\epsilon > 0$ the random measures:*

$$\begin{aligned}M_{\gamma,g,\epsilon}(dx) &= \epsilon^{\frac{\gamma^2}{2}} e^{\gamma X_{g,\epsilon}(x)} d\lambda_g(x), \\ M^\partial_{\gamma,g,\epsilon}(dx) &= \epsilon^{\frac{\gamma^2}{4}} e^{\frac{\gamma}{2}X_{g,\epsilon}(x)} d\lambda_{\partial g}(x).\end{aligned}$$

We then define the random measures $M_{\gamma,g}(dx)$ on Ω and $M^{\partial}_{\gamma,g}(dx)$ on $\partial\Omega$ as the following limits in probability

$$M_{\gamma,g}(dx) := \lim_{\epsilon\to 0} M_{\gamma,g,\epsilon}(dx) = \lim_{\epsilon\to 0} e^{\gamma X_\epsilon(x) - \frac{\gamma^2}{2}\mathbb{E}[X_\epsilon(x)^2]} g(x)^{\frac{\gamma^2}{4}} g_P(x)^{\frac{\gamma^2}{2}} e^{\frac{\gamma^2}{2}h(x)} d\lambda_g(x)$$
$$M^{\partial}_{\gamma,g}(dx) := \lim_{\epsilon\to 0} M^{\partial}_{\gamma,g,\epsilon}(dx) = \lim_{\epsilon\to 0} e^{\frac{\gamma}{2} X_\epsilon(x) - \frac{\gamma^2}{8}\mathbb{E}[X_\epsilon(x)^2]} g(x)^{\frac{\gamma^2}{8}} e^{\frac{\gamma^2}{8}h_\partial(x)} d\lambda_{\partial g}(x)$$

in the sense of weak convergence of measures respectively over Ω and $\partial\Omega$.

In order to prove the Weyl anomaly we will need to write how these measures behave under a conformal change of metric. More precisely for $g' = e^{\varphi} g$ we have:

$$M_{\gamma,g'}(dx) = e^{(1+\frac{\gamma^2}{4})\varphi(x)} M_{\gamma,g}(dx), \qquad (4.2.28)$$
$$M^{\partial}_{\gamma,g'}(dx) = e^{\frac{1}{2}(1+\frac{\gamma^2}{4})\varphi(x)} M^{\partial}_{\gamma,g}(dx).$$

To have a finite partition function (see section 4.3.1), we must show that these measures give an almost surely finite mass to the annulus and to its boundary. Here we write the results in the flat metric $g = dx^2$. We will use the notations M_{γ,dx^2} and $M^{\partial}_{\gamma,dx^2}$ for the bulk and boundary GMC measures in the flat metric. For the boundary measure the expectation of the total mass of $\partial\Omega$ is finite:

$$\mathbb{E}[M^{\partial}_{\gamma,dx^2}(\partial\Omega)] = \int_{\partial\Omega} e^{\frac{\gamma^2}{8}h_\partial(x)} d\lambda_\partial(x) < +\infty.$$

On the other hand for the bulk measure on Ω, this is not straightforward at all as for instance the expectation is infinite as soon as $\gamma^2 \geq 2$ because of the divergence of $g_P(x)$ on the boundary:

$$\mathbb{E}[M_{\gamma,dx^2}(\Omega)] = \int_{\Omega} g_P(x)^{\frac{\gamma^2}{2}} e^{\frac{\gamma^2}{2}h(x)} d\lambda(x) = +\infty \text{ when } \gamma^2 \geq 2.$$

But as it is shown in [40], the random variable $M_{\gamma,dx^2}(\Omega)$ is almost surely finite for all values of $\gamma \in (0,2)$. Therefore we have:

Proposition 4.2.3. *For $\gamma \in (0,2)$ the following quantities are almost surely finite:*

$$M_{\gamma,g}(\Omega) \text{ and } M^{\partial}_{\gamma,g}(\partial\Omega).$$

Finally we state here two very useful tools that we will need for sections 4.3 and 4.4. We start with the Girsanov transform (also called the Cameron-Martin formula) that we will always use in the following way:

Proposition 4.2.4. *(Girsanov transform) Consider the Gaussian free field X on Ω and let $f : \Omega \mapsto \mathbb{R}$ be a continuous function. We introduce the random variable $Y = \int_\Omega f(z)X(z)d\lambda(z)$. Then for any suitable functional F we have:*

$$\mathbb{E}[F((X(z))_{z\in\Omega})e^Y] = e^{\frac{\mathbb{E}[Y^2]}{2}} \mathbb{E}[F((X(z) + \mathbb{E}[X(z)Y])_{z\in\Omega})].$$

The same result holds if we define Y with an integral over the boundary $\partial\Omega$.

We will also need Kahane's convexity inequalities:

Proposition 4.2.5. *(Convexity inequalities, see [42]) Let* $(Y(z))_{z\in\Omega}$ *and* $(Z(z))_{z\in\Omega}$ *be continuous centered Gaussian fields such that*

$$\mathbb{E}[Y(z)Y(z')] \leq \mathbb{E}[Z(z)Z(z')].$$

Then for all convex (resp. concave) functions $F : \mathbb{R}_+ \to \mathbb{R}$ *with at most polynomial growth at infinity*

$$\mathbb{E}\left[F\left(\int_\Omega e^{Y(z)-\frac{\mathbb{E}[Y(z)^2]}{2}} d\lambda(z)\right)\right] \leq (\textit{resp.} \geq)\ \mathbb{E}\left[F\left(\int_\Omega e^{Z(z)-\frac{\mathbb{E}[Z(z)^2]}{2}} d\lambda(z)\right)\right]. \tag{4.2.29}$$

By using our regularization procedure we can apply this result to the case where Y *and* Z *are Gaussian free fields. We also have the same result if we replace the integral over* Ω *by an integral over the boundary* $\partial\Omega$*.*

4.3 Liouville quantum field theory

4.3.1 Defining the partition function

We are now ready to give the expression of the partition function for a metric $g = e^\varphi dx^2$. To write the regularized partition function we must add the renormalization in ϵ of Proposition 4.2.2 whenever there is an exponential of the field $X_{g,\epsilon}$. More explicitly[8]

$$\Pi_{\gamma,\mu,\mu_\partial}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(\epsilon, g, F) = \mathcal{Z}_{GFF}(g) \int_{\mathbb{R}} \mathbb{E}[F(X_{g,\epsilon} + c) \prod_{i=1}^n \epsilon^{\frac{\alpha_i^2}{2}} e^{\alpha_i (X_{g,\epsilon}+c)(z_i)} \prod_{j=1}^{n'} \epsilon^{\frac{\beta_j^2}{4}} e^{\frac{\beta_j}{2}(X_{g,\epsilon}+c)(s_j)}$$

$$\times \exp(-\frac{Q}{4\pi}\int_\Omega R_g(c + X_{g,\epsilon}) d\lambda_g - \mu e^{\gamma c} M_{\gamma,g,\epsilon}(\Omega) - \frac{Q}{2\pi}\int_{\partial\Omega} K_g(c + X_{g,\epsilon}) d\lambda_{\partial g} - \mu_\partial e^{\frac{\gamma}{2}c} M^\partial_{\gamma,g,\epsilon}(\partial\Omega))]dc \tag{4.3.1}$$

Our parameters γ, Q, μ, μ_∂ satisfy $\gamma \in (0, 2)$, $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$, $\mu \geq 0$, $\mu_\partial \geq 0$, and $\mu + \mu_\partial > 0$. The term $\mathcal{Z}_{GFF}(g)$ is the partition function of the GFF coming from (4.2.10). It satisfies (4.5.7), see for instance [23]:

$$\mathcal{Z}_{GFF}(e^\varphi dx^2) = e^{\frac{1}{96\pi}(\int_\Omega |\partial\varphi|^2 d\lambda + 4\int_{\partial\Omega} K\varphi d\lambda_\partial)} \mathcal{Z}_{GFF}(dx^2). \tag{4.3.2}$$

The value of $\mathcal{Z}_{GFF}(dx^2)$ has no importance for the construction of LQFT but it will play a role in the section on LQG as it depends on the outer radius τ of our annulus, see section 4.5.2. The main goal of what follows is to prove that we can let ϵ go to 0 and obtain a finite non zero limit for the partition function:

$$\Pi_{\gamma,\mu,\mu_\partial}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(g, F) = \lim_{\epsilon\to 0} \Pi_{\gamma,\mu,\mu_\partial}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(\epsilon, g, F). \tag{4.3.3}$$

Existence and non-triviality of this limit will depend on the Seiberg bounds:

$$\sum_{i=1}^n \alpha_i + \sum_{j=1}^{n'} \frac{\beta_j}{2} > 0 \tag{4.3.4}$$

$$\forall i,\ \alpha_i < Q \tag{4.3.5}$$

$$\forall j,\ \beta_j < Q \tag{4.3.6}$$

[8]The definition we give is slightly different than the one in [17] or [40]. Instead of adding a factor $\frac{Q}{2}\ln g$ to the field X we include the dependence in g in the regularization procedure of section 4.2.4. This leads to the framework of [36] but it is completely equivalent to the framework of [17, 40].

Theorem 4.3.1. *We have the following alternatives:*

1. *If $\mu > 0$ and $\mu_\partial > 0$, then the partition function converges and is non trivial if and only if (4.3.4) + (4.3.5) + (4.3.6) hold.*

2. *If $\mu > 0$ and $\mu_\partial = 0$ (resp. if $\mu = 0$ and $\mu_\partial > 0$), then the partition function converges and is non trivial if and only if (4.3.4) + (4.3.5) hold (resp. if (4.3.4) + (4.3.6) hold).*

3. *In all other cases, the partition function is worth 0 or $+\infty$.*

The arguments we will use to prove this theorem come from [17, 40]. We will start by studying the flat metric, meaning that $R = 0$, $K = -1$ on $\partial\Omega_1$, $K = \frac{1}{\tau}$ on $\partial\Omega_\tau$, and $\ln g = 0$ (the generalization to all metrics is given by the Weyl anomaly, see section 4.3.2). We are going to apply the Girsanov transform of Proposition 4.2.4 to the insertion points and to the curvature term. The curvature term shifts $X(z)$ by

$$-\frac{Q}{2\pi}\int_{\partial\Omega} K(x)G(x,z)d\lambda_\partial(x) = -\frac{Q}{\tau+1}(\tau\ln(|z|/\tau) + \ln|z|) \tag{4.3.7}$$

and we also get a global factor $\exp(\frac{Q^2}{2}\ln\tau)$. For the insertions we use Proposition 4.2.1 to write on Ω,

$$\begin{aligned}\epsilon^{\frac{\alpha_i^2}{2}}e^{\alpha_i X_\epsilon(z_i)} &= e^{\alpha_i X_\epsilon(z_i)-\frac{\alpha_i^2}{2}\mathbb{E}[X_\epsilon^2(z_i)]}e^{\frac{\alpha_i^2}{2}\mathbb{E}[X_\epsilon^2(z_i)]+\frac{\alpha_i^2}{2}\ln\epsilon} \\ &= e^{\alpha_i X_\epsilon(z_i)-\frac{\alpha_i^2}{2}\mathbb{E}[X_\epsilon^2(z_i)]}e^{\frac{\alpha_i^2}{2}(\ln g_P(z_i)+h(z_i))}(1+o(1)),\end{aligned} \tag{4.3.8}$$

and on the boundary $\partial\Omega$,

$$\begin{aligned}\epsilon^{\frac{\beta_j^2}{4}}e^{\frac{\beta_j}{2}X_\epsilon(s_j)} &= e^{\frac{\beta_j}{2}X_\epsilon(s_j)-\frac{\beta_j^2}{8}\mathbb{E}[X_\epsilon^2(s_j)]}e^{\frac{\beta_j^2}{8}\mathbb{E}[X_\epsilon^2(s_j)]+\frac{\beta_j^2}{4}\ln\epsilon} \\ &= e^{\frac{\beta_j}{2}X_\epsilon(s_j)-\frac{\beta_j^2}{8}\mathbb{E}[X_\epsilon^2(s_j)]}e^{\frac{\beta_j^2}{8}h_\partial(s_j)}(1+o(1)),\end{aligned} \tag{4.3.9}$$

where in both cases the $o(1)$ is deterministic. From this we can easily apply the Girsanov transform to the insertions and get:

$$\begin{aligned}\Pi_{\gamma,\mu,\mu_\partial}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(dx^2,F,\epsilon) =& \mathcal{Z}_{GFF}(dx^2)(\prod_{i=1}^n g_P(z_i)^{\frac{\alpha_i^2}{2}})e^{C(z,s)}\int_{\mathbb{R}} e^{(\sum_i\alpha_i+\sum_j\frac{\beta_j}{2})c}\,\mathbb{E}[F(X_\epsilon+H_\epsilon+c) \\ &\times\exp(-\mu e^{\gamma c}\int_\Omega e^{\gamma H_\epsilon(x)}M_{\gamma,dx^2,\epsilon}(dx) - \mu_\partial e^{\frac{\gamma c}{2}}\int_{\partial\Omega} e^{\frac{\gamma}{2}H_\epsilon(x)}M_{\gamma,dx^2,\epsilon}^\partial(dx))]dc.\end{aligned} \tag{4.3.10}$$

This leads to the following limit for the partition function:

$$\begin{aligned}\Pi_{\gamma,\mu,\mu_\partial}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(dx^2,F) =& \mathcal{Z}_{GFF}(dx^2)(\prod_{i=1}^n g_P(z_i)^{\frac{\alpha_i^2}{2}})e^{C(z,s)}\int_{\mathbb{R}} e^{(\sum_i\alpha_i+\sum_j\frac{\beta_j}{2})c}\,\mathbb{E}[F(X+H+c) \\ &\times\exp(-\mu e^{\gamma c}\int_\Omega e^{\gamma H(x)}M_{\gamma,dx^2}(dx) - \mu_\partial e^{\frac{\gamma c}{2}}\int_{\partial\Omega} e^{\frac{\gamma}{2}H(x)}M_{\gamma,dx^2}^\partial(dx))]dc.\end{aligned} \tag{4.3.11}$$

In the above expressions we have introduced the notations:

$$H(x) = \sum_{i=1}^{n} \alpha_i G(x, z_i) + \sum_{j=1}^{n'} \frac{\beta_j}{2} G(x, s_j) - Q \ln |x| \tag{4.3.12}$$

$$H_\epsilon(x) = \frac{1}{l_\epsilon(x)} \int_{A_\epsilon(x)} H(x+s) ds \tag{4.3.13}$$

$$\begin{aligned} C(z,s) = & \sum_{i<i'} \alpha_i \alpha_{i'} G(z_i, z_{i'}) + \sum_{j<j'} \frac{\beta_j \beta_{j'}}{4} G(s_j, s_{j'}) + \sum_{i,j} \frac{\alpha_i \beta_j}{2} G(z_i, s_j) + \sum_{i=1}^{n} \frac{\alpha_i^2}{2} h(z_i) \\ & + \sum_{j=1}^{n'} \frac{\beta_j^2}{8} h_\partial(s_j) + \frac{Q^2}{2} \ln \tau - \sum_{i=1}^{n} Q \alpha_i \ln |z_i| - \sum_{j=1}^{n'} \frac{Q \beta_j}{2} \ln |s_j| \end{aligned} \tag{4.3.14}$$

Let us now explain why the Seiberg bounds (4.3.4)-(4.3.6) are required to get a non-diverging non-zero limit for (4.3.10). These results are found again in [17, 40].

The first inequality $\sum_i \alpha_i + \sum_j \frac{\beta_j}{2} > 0$ controls the $c \to -\infty$ divergence of the integral. It is required for the regularized partition function (4.3.10) to exist. Let:

$$Z_{0,\epsilon} = \epsilon^{\frac{\gamma^2}{2}} \int_\Omega e^{\gamma(X_\epsilon + H_\epsilon)} d\lambda,$$

$$Z_{0,\epsilon}^\partial = \epsilon^{\frac{\gamma^2}{4}} \int_{\partial\Omega} e^{\frac{\gamma}{2}(X_\epsilon + H_\epsilon)} d\lambda_\partial.$$

We see that $|H_\epsilon| \leq C_\epsilon$ where C_ϵ is a constant depending only on ϵ. Hence we know that $\mathbb{E}[Z_{0,\epsilon}] < \infty$ and thus $Z_{0,\epsilon} < \infty$ almost surely. The same thing holds for $Z_{0,\epsilon}^\partial$. Therefore, we can find $A > 0$ such that $\mathbb{P}(Z_{0,\epsilon} \leq A, Z_{0,\epsilon}^\partial \leq A) > 0$ which gives

$$\Pi_{\gamma,\mu,\mu_\partial}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(dx^2, 1, \epsilon) \geq \tilde{C} \int_{-\infty}^{0} e^{(\sum_i \alpha_i + \sum_j \frac{\beta_j}{2})c} e^{-\mu e^{\gamma c} A - \mu_\partial e^{\frac{\gamma}{2} c} A} \, \mathbb{P}(Z_{0,\epsilon} \leq A, Z_{0,\epsilon}^\partial \leq A) dc$$

for some constant $\tilde{C} > 0$. This last integral diverges if the condition $\sum_i \alpha_i + \sum_j \frac{\beta_j}{2} > 0$ fails to hold. This gives the bounds (4.3.4).

The next thing we need to check is that the following integrals are finite a.s. (otherwise the partition function will be worth 0):

$$\int_\Omega e^{\gamma H(x)} M_{\gamma,dx^2}(dx) \text{ and } \int_{\partial\Omega} e^{\frac{\gamma}{2} H(x)} M_{\gamma,dx^2}^\partial(dx).$$

Proposition 4.2.3 tells us that the integrals without H are finite a.s. This means we can restrict ourselves to looking at what happens around the insertions: this will give the conditions (4.3.5) and (4.3.6). Indeed for a bulk insertion (z_i, α_i), $e^{\gamma H(z)}$ behaves like $\frac{1}{|z-z_i|^{\alpha_i \gamma}}$ around z_i. For $B(z_i, r)$ a small ball around z_i of radius $r > 0$, the results of [17] tell us that:

$$\int_{B(z_i,r)} \frac{1}{|x - z_i|^{\alpha_i \gamma}} M_{\gamma,dx^2}(dx) < +\infty \ a.s. \iff \alpha_i < Q.$$

In the same way we can look at a boundary insertion (s_j, β_j) and we have following [40]:

$$\int_{B(s_j,r)\cap\Omega} \frac{1}{|x - s_j|^{\beta_j \gamma}} M_{\gamma,dx^2}(dx) < +\infty \text{ and } \int_{B(s_j,r)\cap\partial\Omega} \frac{1}{|x - s_j|^{\frac{\beta_j \gamma}{2}}} M_{\gamma,dx^2}^\partial(dx) < +\infty \text{ a.s.} \iff \beta_j < Q.$$

Therefore we have all the conditions of Theorem 4.3.1.

4.3.2 Weyl anomaly

In the previous section we constructed the partition function in the case where the background metric was the flat metric. Here we look at a metric $g = e^{\varphi}dx^2$ and our goal is to find a link between $\Pi_{\gamma,\mu,\mu_\partial}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(g,F)$ and $\Pi_{\gamma,\mu,\mu_\partial}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(dx^2,F)$. We claim that:

Theorem 4.3.2. *(Weyl anomaly) Given a metric $g = e^{\varphi}dx^2$ we have,*[9]

$$\ln \frac{\Pi_{\gamma,\mu,\mu_\partial}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(g,F)}{\Pi_{\gamma,\mu,\mu_\partial}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(dx^2,F(\cdot - \frac{Q}{2}\varphi))} = -\sum_{i=1}^{n}\Delta_{\alpha_i}\varphi(z_i) - \sum_{j=1}^{n'}\frac{1}{2}\Delta_{\beta_j}\varphi(s_j) + \frac{1+6Q^2}{96\pi}\left(\int_\Omega |\partial\varphi|^2 d\lambda + 4\int_{\partial\Omega} K\varphi d\lambda_\partial\right),$$

where we have set $\Delta_{\alpha_i} = \frac{\alpha_i}{2}(Q - \frac{\alpha_i}{2})$ and $\Delta_{\beta_j} = \frac{\beta_j}{2}(Q - \frac{\beta_j}{2})$.

Proof. We need to factor out all the terms containing the metric g in (4.3.1). We start by applying the Girsanov transform to the exponential term:

$$\begin{aligned}&\exp\left(-\frac{Q}{4\pi}\int_\Omega R_g X_{g,\epsilon} d\lambda_g - \frac{Q}{2\pi}\int_{\partial\Omega} K_g X_{g,\epsilon} d\lambda_{\partial g}\right)\\ &= \frac{Q}{4\pi}\int_\Omega \Delta\varphi X_{g,\epsilon} d\lambda - \frac{Q}{2\pi}\int_{\partial\Omega}(K + \frac{1}{2}\partial_n\varphi)X_{g,\epsilon} d\lambda_\partial.\end{aligned}$$

Here we have used (4.2.1) and (4.2.2). Using (4.2.21) this shifts the field $X(x)$ by:

$$\begin{aligned}&\frac{Q}{4\pi}\int_\Omega \Delta\varphi(y)G(x,y)d\lambda(y) - \frac{Q}{2\pi}\int_{\partial\Omega}(K(y) + \frac{1}{2}\partial_n\varphi(y))G(x,y)d\lambda_\partial(y)\\ &= -\frac{Q}{2}(\varphi - m_\partial(\varphi)) - \frac{Q}{2\pi}\int_{\partial\Omega} K(y)G(x,y)d\lambda_\partial(y)\\ &= -\frac{Q}{2}(\varphi(x) - m_\partial(\varphi)) + Q\frac{\ln(1/|x|) + \tau\ln(\tau/|x|)}{\tau+1}.\end{aligned}$$

The term $m_\partial(\varphi)$ does not depend on x and can simply be removed by making a shift on c. The last term $Q\frac{\ln(1/|x|)+\tau\ln(\tau/|x|)}{\tau+1}$ does not depend on the metric g so it will simply cancel out when we take the ratio of the two partition functions. Therefore the only important shift we get is $-\frac{Q}{2}\varphi$. This changes F into $F(\cdot - \frac{Q}{2}\varphi)$ and using the formulas on Gaussian multiplicative chaos (4.2.28) we see that $M_{\gamma,g,\epsilon}$ becomes $M_{\gamma,dx^2,\epsilon}$. Now we look at global factors that appear in front of our partition function. The shift on the insertions gives a term:

$$\prod_{i=1}^{n} e^{-\frac{\alpha_i Q}{2}\varphi(z_i)} \prod_{j=1}^{n'} e^{-\frac{\beta_j Q}{4}\varphi(s_j)}.$$

The metric dependence of our regularization procedure applied to the insertions gives a term:

$$\prod_{i=1}^{n} e^{\frac{\alpha_i^2}{4}\varphi(z_i)} \prod_{j=1}^{n'} e^{\frac{\beta_j^2}{8}\varphi(s_j)}.$$

[9] Our Weyl anomaly coincides with the one of [36] but is slightly different from the one of [17, 40]. This is because we have chosen to include the metric dependence on g in the regularization procedure of section 4.2.4.

We combine the two terms obtained above and get:

$$\prod_{i=1}^{n} e^{-\Delta_{\alpha_i}\varphi(z_i)} \prod_{j=1}^{n'} e^{-\frac{1}{2}\Delta_{\beta_j}\varphi(s_j)}.$$

We now compute the global factor coming from the Girsanov transform. The variance of the curvature term is given by:

$$\begin{aligned}
&\frac{Q^2}{16\pi^2}\int_\Omega\int_\Omega \Delta\varphi(x)\Delta\varphi(y)G(x,y)d\lambda(x)d\lambda(y) + \frac{Q^2}{16\pi^2}\int_{\partial\Omega}\int_{\partial\Omega} \partial_n\varphi(x)\partial_n\varphi(y)G(x,y)d\lambda_\partial(x)d\lambda_\partial(y)\\
&-\frac{Q^2}{8\pi^2}\int_\Omega\int_{\partial\Omega} \Delta\varphi(x)\partial_n\varphi(y)G(x,y)d\lambda(x)d\lambda_\partial(y) + \frac{Q^2}{4\pi^2}\int_{\partial\Omega}\int_{\partial\Omega} K(x)(K(y)+\partial_n\varphi(y))G(x,y)d\lambda_\partial(x)d\lambda_\partial(y)\\
&-\frac{Q^2}{4\pi^2}\int_\Omega\int_{\partial\Omega} \Delta\varphi(x)K(y)G(x,y)d\lambda(x)d\lambda_\partial(y)\\
&= Q^2\ln\tau + \frac{Q^2}{8\pi}\int_\Omega |\partial\varphi|^2 d\lambda + \frac{Q^2}{2\pi}\int_{\partial\Omega}\partial_n\varphi(x)\frac{\ln|x|+\tau\ln(|x|/\tau)}{\tau+1}d\lambda_\partial(x) - \frac{Q^2}{2\pi}\int_\Omega \Delta\varphi(x)\frac{\ln|x|+\tau\ln(|x|/\tau)}{\tau+1}d\lambda(x)
\end{aligned}$$

We apply the Green-Riemann formula (4.2.4) on the last term, noticing that $\Delta\ln|x| = 0$ and that $\partial_n(\frac{\ln|x|+\tau\ln|x/\tau|}{\tau+1}) = K$:

$$-\frac{Q^2}{2\pi}\int_\Omega \Delta\varphi(x)\frac{\ln|x|+\tau\ln(|x|/\tau)}{\tau+1}d\lambda(x) = -\frac{Q^2}{2\pi}\int_{\partial\Omega}\partial_n\varphi(x)\frac{\ln|x|+\tau\ln(|x|/\tau)}{\tau+1}d\lambda_\partial(x) + \frac{Q^2}{2\pi}\int_{\partial\Omega}\varphi K d\lambda.$$

Due to this Girsanov transform, we will have the exponential of the following expression in front of our partition function:

$$\frac{Q^2}{2}\ln\tau + \frac{Q^2}{4\pi}\int_{\partial\Omega}\varphi(x)K(x)d\lambda(x) + \frac{Q^2}{16\pi}\int_\Omega |\partial\varphi(x)|^2 d\lambda(x).$$

We must also add the term coming from $\mathcal{Z}_{GFF}(g)$ (4.3.2):

$$\exp\left(\frac{1}{96\pi}(\int_\Omega |\partial\varphi|^2 d\lambda + 4\int_{\partial\Omega} K\varphi d\lambda_\partial)\right).$$

Putting everything together and taking the ratio of the partition function we arrive at the Weyl anomaly. □

The above result can be easily generalized to the case of two conformally equivalent metrics:

Corollary 4.3.3. *Given two metrics g, g' linked by $g' = e^\varphi g$, we have:*

$$\begin{aligned}
&\ln\frac{\Pi_{\gamma,\mu,\mu_\partial}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(g',F)}{\Pi_{\gamma,\mu,\mu_\partial}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(g,F(\cdot-\frac{Q}{2}\varphi))}\\
&= -\sum_{i=1}^{n}\Delta_{\alpha_i}\varphi(z_i) - \sum_{j=1}^{n'}\frac{1}{2}\Delta_{\beta_j}\varphi(s_j) + \frac{1+6Q^2}{96\pi}\left(\int_\Omega |\partial^g\varphi|^2 d\lambda_g + 4\int_{\partial\Omega} K_g\varphi d\lambda_{\partial g} + 2\int_\Omega R_g\varphi d\lambda_g\right).
\end{aligned}$$

Proof. In this case we compute:

$$\begin{aligned}
&\frac{1+6Q^2}{96\pi}\left(4\int_{\partial\Omega}K(x)\varphi(x)d\lambda_\partial(x)+\int_\Omega|\partial\ln g'(x)|^2d\lambda(x)-\int_\Omega|\partial\ln g(x)|^2d\lambda(x)\right)\\
&=\frac{1+6Q^2}{96\pi}\left(4\int_{\partial\Omega}K(x)\varphi(x)d\lambda_\partial(x)+\int_\Omega|\partial\varphi(x)|^2d\lambda(x)+2\int_\Omega\partial\ln g(x)\cdot\partial\varphi(x)d\lambda(x)\right)\\
&=\frac{1+6Q^2}{96\pi}\left(4\int_{\partial\Omega}K(x)\varphi(x)d\lambda_\partial(x)+\int_\Omega|\partial\varphi(x)|^2d\lambda(x)-2\int_\Omega\Delta\ln g(x)\varphi(x)d\lambda(x)+2\int_{\partial\Omega}\partial_n\ln g(x)\varphi(x)d\lambda(x)\right)\\
&=\frac{1+6Q^2}{96\pi}\left(\int_\Omega|\partial^g\varphi(x)|^2d\lambda_g(x)+4\int_{\partial\Omega}K_g(x)\varphi(x)d\lambda_{\partial g}(x)+2\int_\Omega R_g(x)\varphi(x)d\lambda_g(x)\right).
\end{aligned}$$

□

4.3.3 KPZ formula

The KPZ formula was first introduced in [44] and was studied in a more probabilistic setting in [29]. In our case it consists in finding how the partition function behaves under the action of a conformal automorphism ψ of our annulus meaning that we would like to give a relationship between $\Pi_{\gamma,\mu,\mu_\partial}^{(\psi(z_i),\alpha_i)_i,(\psi(s_j),\beta_j)_j}(dx^2,F)$ and $\Pi_{\gamma,\mu,\mu_\partial}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(dx^2,F)$. The proof we give is much simpler than the one in [17] or [40] as we will just apply Theorem 4.3.2. For this we start by writing the invariance of our partition function under the change of coordinate $z\mapsto\psi(z)$:

$$\Pi_{\gamma,\mu,\mu_\partial}^{(\psi(z_i),\alpha_i)_i,(\psi(s_j),\beta_j)_j}(|\psi'|^2dx^2,F(\cdot\circ\psi))=\Pi_{\gamma,\mu,\mu_\partial}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(dx^2,F).$$

This means that we can apply the Weyl anomaly formula with $\varphi=2\ln|\psi'|$. We compute:

$$\int_\Omega|\partial\varphi|^2d\lambda+4\int_{\partial\Omega}K\varphi d\lambda_\partial=-\int_\Omega\varphi\Delta\varphi d\lambda+\int_{\partial\Omega}\partial_n\varphi d\lambda_\partial+4\int_{\partial\Omega}K\varphi d\lambda_\partial.$$

We recall that ψ is either a rotation, the inversion $z\mapsto\frac{\tau}{z}$, or the composition of both. Since the case of the rotation is trivial will assume that $\psi(z)=\frac{\tau}{z}$. We then have $\psi'(z)=-\frac{\tau}{z^2}$ thus $\varphi(z)=2\ln|\frac{\tau}{z^2}|$. From this we get $\Delta\varphi=0$ and $4K+\partial_n\varphi=0$. Therefore the Weyl anomaly gives:

$$\begin{aligned}
&\Pi_{\gamma,\mu,\mu_\partial}^{(\psi(z_i),\alpha_i)_i,(\psi(s_j),\beta_j)_j}(|\psi'|^2dx^2,F(\cdot\circ\psi))=\\
&\prod_{i=1}^n|\psi'(z_i)|^{2\Delta_{\alpha_i}}\prod_{j=1}^{n'}|\psi'(s_j)|^{\Delta_{\beta_j}}\Pi_{\gamma,\mu,\mu_\partial}^{(\psi(z_i),\alpha_i)_i,(\psi(s_j),\beta_j)_j}(dx^2,F(\cdot\circ\psi+Q\ln|\psi'|)).
\end{aligned}$$

We arrive at:

Theorem 4.3.4. *(KPZ formula) For any conformal automorphism ψ of the annulus Ω, the following holds:*

$$\Pi_{\gamma,\mu,\mu_\partial}^{(\psi(z_i),\alpha_i)_i,(\psi(s_j),\beta_j)_j}(dx^2,F(\cdot\circ\psi+Q\ln|\psi'|))=\prod_{i=1}^n|\psi'(z_i)|^{-2\Delta_{\alpha_i}}\prod_{j=1}^{n'}|\psi'(s_j)|^{-\Delta_{\beta_j}}\Pi_{\gamma,\mu,\mu_\partial}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(dx^2,F).$$

We can also give a similar formula for a conformal change of domain. Let D be a (strict) domain of $\mathbb{C}$ with a smooth boundary and conformally equivalent to our annulus and let $\psi:D\mapsto\Omega$ be a conformal map. We choose insertion points (z_i, α_i) in D and (s_j, β_j) in ∂D. We can define in the same way as (4.3.1) the partition function of LQFT on the domain D:

$$\Pi_{\gamma,\mu,\mu_\partial}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(D,\tilde g,F) = \lim_{\epsilon\to 0} \mathcal{Z}_{GFF}(\tilde g)\int_{\mathbb{R}} dc\,\mathbb{E}[F(X_{\tilde g,\epsilon}+c)\prod_{i=1}^{n}\epsilon^{\frac{\alpha_i^2}{2}}e^{\alpha_i(X_{\tilde g,\epsilon}+c)(z_i)}\prod_{j=1}^{n'}\epsilon^{\frac{\beta_j^2}{4}}e^{\frac{\beta_j}{2}(X_{\tilde g,\epsilon}+c)(s_j)}$$
$$\times\exp(-\frac{Q}{4\pi}\int_D R_{\tilde g}(c+X_{\tilde g,\epsilon})d\lambda_{\tilde g} - \mu e^{\gamma c}M_{\gamma,\tilde g,\epsilon}(D) - \frac{Q}{2\pi}\int_{\partial D} K_{\tilde g}(c+X_{\tilde g,\epsilon})d\lambda_{\partial\tilde g} - \mu_\partial e^{\frac{\gamma}{2}c}M^\partial_{\gamma,\tilde g,\epsilon}(\partial D))] \tag{4.3.15}$$

Here $\tilde g$ denotes a metric on D and $X_{\tilde g}$ is the GFF on D with Neumann boundary conditions and with vanishing mean over ∂D in the metric $\tilde g$. We state the following result:

Proposition 4.3.5. *(Conformal change of domain) Let D be a domain of $\mathbb{C}$ with a smooth boundary and conformally equivalent to our annulus Ω. Let $\psi : D \mapsto \Omega$ be a conformal map between D and Ω and let $g_\psi = |\psi'|^2 g(\psi)$ be the pull-back of a metric g on Ω by ψ. Then we have:*

$$\Pi_{\gamma,\mu,\mu_\partial}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(D,g_\psi,F) = \prod_{i=1}^{n}|\psi'(z_i)|^{2\Delta_{\alpha_i}}\prod_{j=1}^{n'}|\psi'(s_j)|^{\Delta_{\beta_j}}\Pi_{\gamma,\mu,\mu_\partial}^{(\psi(z_i),\alpha_i)_i,(\psi(s_j),\beta_j)_j}(\Omega,g,F(\cdot\circ\psi+Q\ln|\psi'|)).$$

4.3.4 Liouville field and measure at fixed τ

Using our partition function we can now give the definition of the Liouville field ϕ_τ, the bulk Liouville measure Z_τ, and the boundary Liouville measure Z^∂_τ.[10] These objects can be defined once the Seiberg bounds (4.3.4)-(4.3.6) are satisfied. Formally, ϕ_τ is the log-conformal factor of the formal random metric $e^{\gamma\phi_\tau}g$ conformally equivalent to g.[11] The Liouville measure is a random measure that can be seen as the volume form of this formal metric tensor whereas the Liouville boundary measure corresponds to the line element along the boundary. More rigorously, given a measured space E, we denote $R(E)$ the space of Radon measures on E equipped with the topology of weak convergence. For any metric g, the joint law of $(\phi_\tau, Z_\tau, Z^\partial_\tau)$ is defined for continuous bounded functionals F on $H^{-1}(\Omega)\times R(\Omega)\times R(\partial\Omega)$ by:

$$\mathbb{E}_{\gamma,\mu,\mu_\partial,g}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}[F(\phi_\tau,Z_\tau,Z^\partial_\tau)] = (\Pi_{\gamma,\mu,\mu_\partial}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(g,1))^{-1}\mathcal{Z}_{GFF}(g)$$
$$\times\lim_{\epsilon\to 0}\int_{\mathbb{R}}\mathbb{E}[F(X_{g,\epsilon}+c,\epsilon^{\frac{\gamma^2}{2}}e^{\gamma(X_{g,\epsilon}+c)}d\lambda_g,\epsilon^{\frac{\gamma^2}{4}}e^{\frac{\gamma}{2}(X_{g,\epsilon}+c)}d\lambda_{\partial g})\prod_i\epsilon^{\frac{\alpha_i^2}{2}}e^{\alpha_i(X_{g,\epsilon}+c)(z_i)}\prod_j\epsilon^{\frac{\beta_i^2}{4}}e^{\frac{\beta_j}{2}(X_{g,\epsilon}+c)(s_j)}$$
$$\times\exp(-\frac{Q}{4\pi}\int_\Omega R_g(c+X_{g,\epsilon})d\lambda_g - \mu e^{\gamma c}M_{\gamma,g,\epsilon}(\Omega) - \frac{Q}{2\pi}\int_{\partial\Omega}K_g(c+X_{g,\epsilon})d\lambda_{\partial g} - \mu_\partial e^{\frac{\gamma}{2}c}M^\partial_{\gamma,g,\epsilon}(\partial\Omega))]dc.$$

We can write a more compact expression for the joint law of the bulk and boundary measures in the case where $g = dx^2$. We introduce the notations,

$$Z_0(dx) = e^{\gamma H(x)}M_{\gamma,dx^2}(dx),$$
$$Z_0^\partial(dx) = e^{\frac{\gamma}{2}H(x)}M^\partial_{\gamma,dx^2}(dx),$$

[10] Here we write the subscript τ to emphasize the fact that we are working at fixed τ, i.e. on the annulus Ω of radii 1 and τ. In section 4.4 we will integrate over $\tau\in(1,\infty)$ and define the general Liouville measures Z and Z^∂.

[11] The formal random metric $e^{\gamma\phi_\tau}g$ has been constructed rigorously by Miller-Sheffield for the special value $\gamma=\sqrt{\frac{8}{3}}$ corresponding to uniform planar maps, see for instance [56].

and the ratio $R = \frac{Z_0(\Omega)}{Z_0^\partial(\partial\Omega)^2}$. Then we get for the law of the bulk and boundary measures in the flat metric:

$$\begin{aligned}
&\mathbb{E}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}_{\gamma,\mu,\mu_\partial,dx^2}[F(Z_\tau, Z_\tau^\partial)] \\
&= \frac{1}{\mathcal{Z}}\int_{\mathbb{R}} e^{(\sum_i \alpha_i + \frac{1}{2}\sum_j \beta_j)c}\mathbb{E}[F(e^{\gamma c}Z_0, e^{\frac{\gamma}{2}c}Z_0^\partial)\exp(-\mu e^{\gamma c}Z_0(\Omega) - \mu_\partial e^{\frac{\gamma}{2}c}Z_0^\partial(\partial\Omega))]dc \\
&= \frac{1}{\mathcal{Z}}\int_0^\infty y^{\frac{2}{\gamma}(\sum_i \alpha_i + \frac{1}{2}\sum_j \beta_j)-1}\mathbb{E}[F(y^2R\frac{Z_0}{Z_0(\Omega)}, y\frac{Z_0^\partial}{Z_0^\partial(\partial\Omega)})\exp(-\mu y^2 R - \mu_\partial y)Z_0^\partial(\partial\Omega)^{-\frac{2}{\gamma}(\sum_i \alpha_i + \frac{1}{2}\sum_j \beta_j)}]dy.
\end{aligned}$$

The normalization constant $\mathcal{Z}$ is computed by choosing $F = 1$. We can get an even simpler expression if we choose $\mu_\partial = 0$, in this case we have:

Corollary 4.3.6. *Assume $\mu_\partial = 0$. The joint law of the bulk/boundary measures are given by*

$$\begin{aligned}
&\mathbb{E}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}_{\gamma,\mu,\mu_\partial=0,dx^2}[F(Z_\tau, Z_\tau^\partial)] = \\
&\frac{1}{\mathcal{Z}}\int_0^\infty u^{\frac{1}{\gamma}(\sum_i \alpha_i + \frac{1}{2}\sum_j \beta_j)-1}\mathbb{E}[F(u\frac{Z_0}{Z_0(\Omega)}, u^{1/2}\frac{Z_0^\partial}{Z_0(\Omega)^{1/2}})Z_0(\Omega)^{-\frac{1}{\gamma}(\sum_i \alpha_i + \frac{1}{2}\sum_j \beta_j)}]e^{-\mu u}du
\end{aligned}$$

where $\mathcal{Z}$ is the correct normalization constant in order to get a probability measure. In particular, the law of the volume of Ω follows a Gamma law with parameters $(\frac{\sum_i \alpha_i + \frac{1}{2}\sum_j \beta_j}{\gamma}, \mu)$.

A similar expression holds for $\mu = 0$.

4.4 Liouville quantum gravity

4.4.1 Convergence of the partition function

We are now ready to construct the full theory of Liouville quantum gravity (LQG). Our goal is to integrate over τ the partition function of LQFT constructed in section 4.3 with an appropriate measure. As explained in section 4.5.1 of the appendix, the correct partition function of LQG is given by,

$$\mathcal{Z}_{LQG} = \int_1^\infty d\tau \mathcal{Z}_{Ghost}(\tau, dx^2)\mathcal{Z}_{Matter}(\tau, dx^2)\mathcal{Z}_{LQFT}(\tau, dx^2), \tag{4.4.1}$$

where we have following expressions for the different partition functions,[12]

$$\begin{aligned}
\mathcal{Z}_{GFF}(\tau, dx^2) &= \tau^{1/12}\frac{1}{|\eta(\tau)|} \\
\mathcal{Z}_{Matter}(\tau, dx^2) &= (\mathcal{Z}_{GFF})^{c_m} = \tau^{c_m/12}\frac{1}{|\eta(\tau)|^{c_m}} \\
\mathcal{Z}_{Ghost}(\tau, dx^2) &= \frac{1}{\tau}\tau^{-\frac{13}{6}}|\eta(\tau)|^2 \\
\mathcal{Z}_{LQFT}(\tau, dx^2) &= \int_{\partial\Omega} d\lambda_\partial(s)\Pi^{(s,\gamma)}_{\gamma,\mu,\mu_\partial}(\tau, dx^2),
\end{aligned}$$

[12]In the section we modify very slightly the notation for the partition function of LQFT. Here we always choose $F = 1$ and remove it from the notation but we include τ to show explicitly the dependence on the modulus.

with

$$\eta(\tau) = \tau^{-1/12} \prod_{n=1}^{\infty} (1 - \tau^{-2n}). \tag{4.4.2}$$

The real constant c_m is the central charge of the matter fields. It is link to Q by the relation written in the appendix (4.5.8):

$$c_m - 25 + 6Q^2 = 0. \tag{4.4.3}$$

Let us explain the expression for $\mathcal{Z}_{LQFT}$. It corresponds to the partition function of LQFT that we have constructed in section 4.3 with one insertion point on the boundary (this is the minimal condition required for the Seiberg bounds (4.3.4)-(4.3.6) to be satisfied). We choose an insertion point on the boundary $s \in \partial\Omega$ with weight γ and we integrate the position of s over the whole boundary. This is the correct choice to get a link with random planar maps, see section 4.4.3. From all of this we get:

$$\begin{aligned} \mathcal{Z}_{LQG} &= \int_{\partial\Omega} d\lambda_\partial(s) \int_1^\infty d\tau \mathcal{Z}_{Ghost}(\tau, dx^2) \mathcal{Z}_{Matter}(\tau, dx^2) \Pi^{(s,\gamma)}_{\gamma,\mu,\mu_\partial}(\tau, dx^2) \\ &= \int_{\partial\Omega} d\lambda_\partial(s) \int_1^\infty \frac{d\tau}{\tau} \tau^{\frac{c_m-25}{12}} |\eta(\tau)|^{1-c_m} \mathcal{Z}_{GFF}^{-1} \Pi^{(s,\gamma)}_{\gamma,\mu,\mu_\partial}(\tau, dx^2). \end{aligned} \tag{4.4.4}$$

The integral over the position of the boundary insertion can be easily simplified using the KPZ formula of section 4.3.3. For any $s \in \partial\Omega$, we can write $s = \psi(1)$ for a conformal automorphism ψ. A computation gives:

$$\int_{\partial\Omega} d\lambda_\partial(s) \Pi^{(s,\gamma)}_{\gamma,\mu,\mu_\partial}(\tau, dx^2) = 2\pi(1 + \tau^{1-\frac{\gamma}{2}(Q-\frac{\gamma}{2})}) \Pi^{(1,\gamma)}_{\gamma,\mu,\mu_\partial}(\tau, dx^2) = 4\pi \Pi^{(1,\gamma)}_{\gamma,\mu,\mu_\partial}(\tau, dx^2). \tag{4.4.5}$$

In the following we will drop the irrelevant factor 4π. Our goal is now to prove:

Theorem 4.4.1. *Let us assume that $\mu_\partial > 0$. Then the partition function of Liouville quantum gravity given by*

$$\mathcal{Z}_{LQG} = \int_1^\infty \frac{d\tau}{\tau} \tau^{\frac{c_m-25}{12}} |\eta(\tau)|^{1-c_m} \mathcal{Z}_{GFF}^{-1} \Pi^{(1,\gamma)}_{\gamma,\mu,\mu_\partial}(\tau, dx^2)$$

is finite for any value of $\gamma \in (0, 2)$.

Proof. Let us first note that thanks to relation (4.4.3) $\gamma < 2$ is equivalent to $c_m < 1$. We will look separately at the behaviour of our integral for $\tau \to \infty$ and for $\tau \to 1$. We write $\mathcal{Z}_{LQG} = \mathcal{Z}^1_{LQG} + \mathcal{Z}^\infty_{LQG}$ with:

$$\begin{aligned} \mathcal{Z}^1_{LQG} &= \int_1^2 \frac{d\tau}{\tau} \tau^{\frac{c_m-25}{12}} |\eta(\tau)|^{1-c_m} \mathcal{Z}_{GFF}^{-1} \Pi^{(1,\gamma)}_{\gamma,\mu,\mu_\partial}(\tau, dx^2), \\ \mathcal{Z}^\infty_{LQG} &= \int_2^\infty \frac{d\tau}{\tau} \tau^{\frac{c_m-25}{12}} |\eta(\tau)|^{1-c_m} \mathcal{Z}_{GFF}^{-1} \Pi^{(1,\gamma)}_{\gamma,\mu,\mu_\partial}(\tau, dx^2). \end{aligned}$$

Throughout this proof we will use the abuse of notation $e^{\gamma X(x)} d\lambda(x)$ and $e^{\frac{\gamma}{2} X(x)} d\lambda_\partial(x)$ for $M_{\gamma,dx^2}(dx)$ and $M^\partial_{\gamma,dx^2}(dx)$. Using (4.3.11) we can write,

$$\frac{\Pi^{(1,\gamma)}_{\gamma,\mu,\mu_\partial}(\tau, dx^2)}{\mathcal{Z}_{GFF}(\tau, dx^2)} = e^{C(1)} \int_{\mathbb{R}} e^{\frac{\gamma c}{2}} \mathbb{E}[\exp(-\mu e^{\gamma c} \int_\Omega e^{\gamma H + \gamma X} d\lambda - \mu_\partial e^{\frac{\gamma c}{2}} \int_{\partial\Omega} e^{\gamma H/2 + \gamma X/2} d\lambda_\partial)] dc,$$

with:

$$H(x) = \frac{\gamma}{2} G(x,s) - Q \ln |x|,$$
$$C(s) = \frac{\gamma^2}{8} h_\partial(s) + \frac{Q^2}{2} \ln \tau - \frac{Q\gamma}{2} \ln |s|.$$

Therefore

$$C(1) = \frac{\gamma^2}{8} h_\partial(1) + \frac{Q^2}{2} \ln \tau = \frac{Q^2}{2} \ln \tau + \frac{\gamma^2}{8} \left(\frac{\tau^2 \ln \tau}{(1+\tau)^2} + \sum_{n=1}^{\infty} \frac{\tau^{2n}+1}{n\tau^{2n}(\tau^{2n}-1)} + \ln \frac{\tau^4}{(\tau^2-1)^2} \right) + \tilde{C}$$

where $\tilde{C} > 0$ is some constant. We restrict ourselves to the integral over the inner boundary $\partial\Omega_1$ and we use the following upper bound:

$$\begin{aligned} \frac{\Pi^{(1,\gamma)}_{\gamma,\mu,\mu_\partial}(\tau, dx^2)}{\mathcal{Z}_{GFF}(\tau, dx^2)} &\leq e^{C(1)} \int_{\mathbb{R}} e^{\frac{\gamma c}{2}} \mathbb{E}[\exp(-\mu_\partial e^{\frac{\gamma c}{2}} \int_{\partial\Omega_1} e^{\gamma H/2 + \gamma X/2} d\lambda_\partial)] dc \\ &\leq e^{C(1)} \frac{2}{\gamma} \frac{1}{\mu_\partial} \mathbb{E}\left[\frac{1}{\int_{\partial\Omega_1} e^{\gamma H/2+\gamma X/2} d\lambda_\partial} \right]. \end{aligned}$$

We will decompose our process X into two parts Y and Z according to the following decomposition of the Green's function, written for $z = e^{i\theta}$, $z' = e^{i\theta'}$:

$$G_Y(z,z') = g_0(1,1) + 2\sum_{n=1}^{\infty} g_n(1,1) \cos n(\theta-\theta') = \frac{\tau^2 \ln \tau}{(1+\tau)^2} + \sum_{n=1}^{+\infty} \frac{\tau^{2n}+1}{n\tau^{2n}(\tau^{2n}-1)} \cos n(\theta - \theta'),$$
$$G_Z(z,z') = \ln \frac{|\tau^4 z^2 z'^2|}{|1-z\overline{z'}||\tau^2 - z\overline{z'}||z-z'||\tau^2 z - z'|} = \ln \frac{|\tau^4|}{|e^{i\theta} - e^{i\theta'}|^2 |\tau^2 e^{i\theta} - e^{i\theta'}|^2}.$$

Step 1: $\tau \to \infty$

We start by looking at the behaviour of our integral when $\tau \to \infty$. We see a diverging term $\frac{\tau^2 \ln \tau}{(1+\tau)^2}$ in our correlation function G_Y but this is simply a constant which means it corresponds to a constant Gaussian variable independent of everything else. We can remove it simply by making a shift on c in (4.3.11). This also removes the corresponding term in $C(1)$. For the remaining part of G_Y, we have uniformly on $\partial\Omega_1$:

$$\sum_{n=1}^{+\infty} \frac{\tau^{2n}+1}{n\tau^{2n}(\tau^{2n}-1)} \cos n(\theta-\theta') \underset{\tau\to\infty}{\to} 0.$$

For G_Z we have in the same way, uniformly on $\partial\Omega_1$:

$$\ln \frac{|\tau^4|}{|\tau^2 e^{i\theta} - e^{i\theta'}|^2} \underset{\tau\to\infty}{\to} 0.$$

This means that by using Kahane's inequalities (4.2.29) we can bound these terms that converge to 0 and we are left only with $\ln \frac{1}{|z-z'|^2}$ in our covariance function. We call $\hat{Z}$ a Gaussian process of covariance $\ln \frac{1}{|z-z'|^2}$. Similarly for $C(1)$, as $\tau \to \infty$ the only terms that remain are $\frac{Q^2}{2} \ln \tau$ and the constant $\tilde{C} > 0$. Concerning $\eta(\tau)$ we bound it in this case by $\frac{1}{\tau^{12}}$. Combining everything we get, for some $C_1 > 0$:

$$\mathcal{Z}_{LQG}^{\infty} \leq C_1 \int_2^{\infty} d\tau \tau^{\frac{c_m - 25 + 6Q^2}{12}} \frac{1}{\tau^{1+\frac{1-c_m}{12}}} \mathbb{E}\left[\frac{1}{\int_{\partial\Omega_1} e^{\frac{\gamma}{2}\hat{Z}(x)} |x-1|^{-\frac{\gamma^2}{2}} d\lambda_\partial(x)}\right].$$

The theory of Gaussian multiplicative chaos [72] tells us that the above expectation $\mathbb{E}[\cdot]$ is finite and it is clearly independent of τ. Using the relation (4.4.3) we can simplify the powers of τ: we are left with $\tau^{-1-\frac{1-c_m}{12}}$ which is integrable for $c_m < 1$. Therefore the integral converges for $\tau \to \infty$.

Step 2: $\tau \to 1$

In this case we have to be more careful as G_Y has a divergence in τ that depends on z. Indeed the sum $\sum_{n=1}^{+\infty} \frac{\tau^{2n}+1}{n\tau^{2n}(\tau^{2n}-1)}$ diverges as $\frac{\pi^2}{6}\frac{1}{\tau-1}$ as $\tau \to 1$. Therefore instead of integrating over the full inner boundary $\partial\Omega_1$, we will restrict ourselves to a small open neighbourhood $V(\epsilon)$ of 1 such that for $x, y \in V(\epsilon)$, $|G_Y(x,y) - G_Y(1,1)| \leq \epsilon\, G_Y(1,1)$ for a fixed $\epsilon > 0$. For this bound to hold the size of $V(\epsilon)$ needs to scale as $\ln\tau$ with τ. We then get:

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_{\partial\Omega_1} e^{\frac{\gamma}{2}(Y(x)+Z(x)) + \frac{\gamma^2}{4}(G_Y(x,1)+G_Z(x,1))} d\lambda_\partial(x)\right)^{-1}\right] \\
&\leq \mathbb{E}\left[\left(\int_{V(\epsilon)} e^{\frac{\gamma}{2}(Y(x)+Z(x)) + \frac{\gamma^2}{4}(G_Y(x,1)+G_Z(x,1))} d\lambda_\partial(x)\right)^{-1}\right] \\
&\leq e^{-\frac{\gamma^2}{4}(1-\epsilon)G_Y(1,1)} \mathbb{E}\left[\left(\int_{V(\epsilon)} e^{\frac{\gamma}{2}(Y(x)+Z(x)) + \frac{\gamma^2}{4}G_Z(x,1)} d\lambda_\partial(x)\right)^{-1}\right] \\
&\leq e^{-\frac{\gamma^2}{4}(1-\epsilon)G_Y(1,1) + \frac{\gamma^2}{8}(1+\epsilon)G_Y(1,1)} \mathbb{E}\left[\left(\int_{V(\epsilon)} e^{\frac{\gamma}{2}Z(x) + \frac{\gamma^2}{4}G_Z(x,1)} d\lambda_\partial(x)\right)^{-1}\right].
\end{aligned}$$

In the last inequality we have used Kahane's inequalities (4.2.29) to replace the $e^{\frac{\gamma}{2}Y(x)}$ in the denominator by $e^{\frac{\gamma}{2}\sqrt{1+\epsilon}Y(1)}$. For the covariance G_Z, the log term $\ln\frac{1}{|\tau^2 e^{i\theta} - e^{i\theta'}|^2}$ also diverges as $\tau \to 1$. We will simply bound it by $\ln\frac{1}{|\tau^2-1|^2}$. Using (4.2.29) this gives a polynomial divergence which is irrelevant compared to the exponential terms. Finally there is a last exponential divergence coming from the constant $C(1)$ which is of the order of $e^{\frac{\gamma^2}{8}G_Y(1,1)}$. Calling again $\hat{Z}$ a Gaussian process of covariance $\ln\frac{1}{|z-z'|^2}$, we get for some $C_2 > 0$:

$$\mathcal{Z}_{LQG}^{1} \leq C_2 \int_1^2 d\tau |\eta(\tau)|^{1-c_m} e^{\frac{3\epsilon\gamma^2}{8}G_Y(1,1)} \mathbb{E}\left[\left(\int_{V(\epsilon)} e^{\frac{\gamma}{2}\hat{Z}(x)} |x-1|^{-\frac{\gamma^2}{2}} d\lambda_\partial(x)\right)^{-1}\right].$$

To finish we use the following bound on $\eta(\tau)$:

$$\eta(\tau) = \tau^{-1/12} \prod_{n=1}^{\infty} (1-\tau^{-2n}) \leq \tau^{-1/12} e^{-\frac{1}{\tau^2-1}}.$$

Therefore we get for some $C_3 > 0$:

$$\mathcal{Z}_{LQG}^{1} \leq C_3 \int_1^2 d\tau e^{(\frac{\pi^2\gamma^2}{16}\epsilon - (\frac{1-c_m}{2}))\frac{1}{\tau-1}} \mathbb{E}\left[\left(\int_{V(\epsilon)} e^{\frac{\gamma}{2}\hat{Z}(x)} |x-1|^{-\frac{\gamma^2}{2}} d\lambda_\partial(x)\right)^{-1}\right].$$

$V(\epsilon)$ scales as $\ln\tau$ with τ so the above expectation $\mathbb{E}[\cdot]$ produces a logarithmic divergence which is again negligible compared to the exponential terms (see for instance [73]). Since $c_m < 1$, this quantity is bounded if we choose ϵ small enough. Therefore $\mathcal{Z}_{LQG} < +\infty$. □

4.4.2 Joint law of the Liouville measures and of the random modulus

Our goal in this section is to give the joint law of the bulk Liouville measure Z, the boundary Liouville measure Z^∂, and of the random τ[13]. Because the annulus $\Omega = \{z, 1 < |z| < \tau\}$ depends explicitly on τ, when we perform an integration on τ this poses a problem to have a fix domain on which to define Z and Z^∂. To solve this problem we will work on a fix annulus $\hat{\Omega} = \{z, 1 < |z| < 2\}$ which is simply the annulus of radii 1 and 2. We then introduce the map $f_\tau : \hat{\Omega} \mapsto \Omega$ with expression given in polar coordinates $z = re^{i\theta}$ by:

$$f_\tau(re^{i\theta}) = ((\tau-1)(r-1)+1)e^{i\theta}. \tag{4.4.6}$$

Of course f_τ will not be a conformal map. For any Borel set $A \subset \hat{\Omega}$, we will write $f_\tau(A)$ for its image under f_τ (and similarly for subsets of $\partial\hat{\Omega}$). Just like in the previous subsection we will work with one insertion point of weight γ placed at the point 1 on the inner boundary of $\hat{\Omega}$ (but this choice is irrelevant thanks to the KPZ relation). We are now ready to give the expression of the joint law of (Z, Z^∂, τ). For any Borel sets $A \subset \hat{\Omega}$, $B \subset \partial\hat{\Omega}$ and for any continuous bounded functionals F we have:

$$\begin{aligned}\mathbb{E}^\gamma_{\mu,\mu_\partial}[F(Z(A), Z^\partial(B), \tau)] = \frac{1}{\mathcal{Z}}\int_1^\infty d\tau \tau^{\frac{c_m-25}{12}-1}|\eta(\tau)|^{1-c_m} \\ \times \int_0^\infty dy \mathbb{E}[F(y^2 R\frac{Z_0(f_\tau(A))}{Z_0(\Omega)}, y\frac{Z_0^\partial(f_\tau(B))}{Z_0^\partial(\partial\Omega)}, \tau)\exp(-\mu y^2 R - \mu_\partial y) Z_0^\partial(\partial\Omega)^{-1}].\end{aligned}$$

In this expression and in the following ones the normalization constant $\mathcal{Z}$ is always computed by choosing $F = 1$. The definitions of Z_0, Z_0^∂ and R are given in section 4.3.4. By applying the above formula we can give the joint law of the total inner boundary, total outer boundary and of the random τ:

$$\begin{aligned}\mathbb{E}^\gamma_{\mu,\mu_\partial}[F(Z^\partial(\partial\hat{\Omega}_1), Z^\partial(\partial\hat{\Omega}_2), \tau)] = \frac{1}{\mathcal{Z}}\int_1^\infty d\tau \tau^{\frac{c_m-25}{12}-1}|\eta(\tau)|^{1-c_m} \\ \times \int_0^\infty dy \mathbb{E}[F(y\frac{Z_0^\partial(\partial\Omega_1)}{Z_0^\partial(\partial\Omega)}, y\frac{Z_0^\partial(\partial\Omega_\tau)}{Z_0^\partial(\partial\Omega)}, \tau)\exp(-\mu y^2 R - \mu_\partial y) Z_0^\partial(\partial\Omega)^{-1}].\end{aligned}$$

Simplifying the above we get the law of τ:

$$\mathbb{E}^\gamma_{\mu,\mu_\partial}[F(\tau)] = \frac{1}{\mathcal{Z}}\int_1^\infty d\tau \tau^{\frac{c_m-25}{12}-1}|\eta(\tau)|^{1-c_m}\int_0^\infty dy \mathbb{E}[F(\tau)\exp(-\mu y^2 R - \mu_\partial y) Z_0^\partial(\partial\Omega)^{-1}].$$

We can also give a joint law conditioned on the total boundary length $Z^\partial(\partial\hat{\Omega})$, for some $L > 0$:

$$\begin{aligned}&\mathbb{E}^\gamma_{\mu,\mu_\partial}[F(Z(A), Z^\partial(B), \tau)|Z^\partial(\partial\hat{\Omega}) = L] = \\ &\frac{1}{\mathcal{Z}}\int_1^\infty d\tau \tau^{\frac{c_m-25}{12}-1}|\eta(\tau)|^{1-c_m}\mathbb{E}[F(L^2 R\frac{Z_0(f_\tau(A))}{Z_0(\Omega)}, L\frac{Z_0^\partial(f_\tau(B))}{Z_0^\partial(\partial\Omega)}, \tau)\exp(-\mu L^2 R - \mu_\partial L) Z_0^\partial(\partial\Omega)^{-1}].\end{aligned}$$

If we were to condition on τ we would recover the formulas of section 4.3.4.

[13] In the literature the modulus of an annulus of radii 1 and τ is given by $\frac{1}{2\pi}\ln\tau$ but here we choose to write everything with our parameter τ.

4.4.3 Conjectured link with random planar maps

The theory of Liouville quantum gravity that we have constructed is the conjectured limit of random planar maps potentially weighted by a certain statistical physics model. Let us try to state a precise mathematical conjecture. We call $\mathcal{Q}_{n,p}$ the set of quadrangulations with the topology of an annulus, n inner faces, a perimeter of $2p$ and with one marked point on the boundary. The growth of $\mathcal{Q}_{n,p}$ is expected to be of the order $\mu_c^n \mu_{\partial c}^{2p}$ times a polynomial term. The constants μ_c and $\mu_{\partial c}$ are non universal and they depend on the discrete model chosen (the values change for instance for triangulations). Now we fix an $a > 0$ and introduce $\overline{\mu} = \mu_c + a^2\mu$, $\overline{\mu}_\partial = \mu_{\partial c} + a\mu_\partial$. We can give a conformal structure to each $Q \in \mathcal{Q}_{n,p}$ and map it to an annulus of radii 1 and τ for some unique τ where the marked point of Q is mapped to 1. Then by using the function f_τ (4.4.6) we can map Q to the reference annulus $\hat{\Omega}$ defined in the previous subsection. By using this mapping, for each $Q \in \mathcal{Q}_{n,p}$ we define the bulk and boundary measures $\nu_{Q,a}$ and $\nu_{Q,a}^\partial$ on $\hat{\Omega}$ by giving a volume a^2 to each face of Q and a length a to each edge on the boundary. We now define the random measures (ν_a, ν_a^∂) by the relation, for all suitable functionals F:

$$\mathbb{E}^a[F(\nu_a, \nu_a^\partial)] := \frac{1}{\mathcal{Z}_a} \sum_{N,p} e^{-\overline{\mu}N} e^{-\overline{\mu}_\partial 2p} \sum_{Q \in \mathcal{Q}_{N,p}} F(\nu_{Q,a}, \nu_{Q,a}^\partial).$$

The constant $\mathcal{Z}_a$ is again the normalization constant. We now state:

Conjecture 11. *The limit in law when $a \to 0$ of (ν_a, ν_a^∂) exists in the space of Radon measures equipped with the topology of weak convergence and is given by the measures Z and Z^∂ with $\gamma = \sqrt{\frac{8}{3}}$ and with appropriate cosmological constants μ and μ_∂. More precisely we have*

$$\lim_{a \to 0} \mathbb{E}^a[F(\nu_a, \nu_a^\partial)] = \mathbb{E}_{\mu,\mu_\partial}^{\sqrt{\frac{8}{3}}}[F(Z, Z^\partial)]$$

for a certain value of μ and μ_∂.

If we couple the planar maps to a statistical physics model, a similar conjecture is expected to hold but for a different value of γ (see [18] for a discussion of this in the case of the torus).

4.5 Appendix

4.5.1 Liouville quantum gravity and conformal field theory

Let us give some heuristic ideas on the theory of 2D quantum gravity and on conformal field theory. For a two-dimensional surface M let $\mathcal{M}$ be the set of (smooth) Riemannian metrics on M. The partition function of (euclidean) 2D quantum gravity on the surface M is given by:

$$\mathcal{Z}_{LQG} = \int_{\mathcal{M}} Dg e^{-S_{EH}(g) - S_{Matter}(g)}. \tag{4.5.1}$$

Here S_{EH} is Einstein-Hilbert action with a cosmological constant $\mu_0 > 0$

$$S_{EH}(g) = \int_M d\lambda_g (R_g + \mu_0) \tag{4.5.2}$$

where R_g stands for the scalar curvature of the metric g. S_{Matter} is the action of the matter fields of the theory, we will discuss its expression below. Dg is a formal uniform measure on the space $\mathcal{M}$ and a major difficulty is precisely to give sense to this measure. In two dimensions there are several simplifications. First of all, due to the Gauss-Bonnet theorem we have $\int_M d\lambda_g R_g = 8\pi(1-h)$ where h is the genus of the surface M meaning that this term can be factored out of the integration over $\mathcal{M}$. Next we have a particularly simple description of the space $\mathcal{M}$ in two-dimensions. It turns out that any $g \in \mathcal{M}$ can be written in the simple form $g = \psi(e^{\phi}\hat{g}_\tau)$, where ψ is diffeomorphism (a change of coordinates), $\phi : M \to \mathbb{R}$ is a function called the Weyl factor, and $\hat{g}_\tau$ are a set a reference metrics parametrized by τ which correspond to the moduli space of M (the set of non-equivalent conformal structures on M). The description of this moduli space can be quite complicated for M of arbitrary topology but it remains finite dimensional. With this result in mind it is natural to want to write the formal measure Dg in the following way

$$Dg = D\psi D\phi D\tau J(\phi,\tau) \tag{4.5.3}$$

where $J(\phi,\tau)$ is the infinite dimensional Jacobian of this change of variable. Considering that S_{EH} and S_{Matter} are independent of the choice of coordinates (like any relevant quantity in physics), and so is the Jacobian J, we can simply drop the integration $D\psi$ over the diffeomorphisms. $D\tau$ is the finite dimensional integration measure over the moduli space so it has a well defined expression. We now come to the difficult problem of giving the expression for $D\phi J(\phi,\tau)$. It is heuristically shown in the physics literature [16], [20], [21] that this quantity can be expressed using the Liouville action. We introduce

$$\mathcal{Z}_{LQFT}(\hat{g}_\tau) = \int_\Sigma D_{\hat{g}_\tau} X e^{-S_L(X,\hat{g}_\tau)} \tag{4.5.4}$$

where again $D_{\hat{g}_\tau}X$ is a formal uniform measure on the space Σ of functions $X : M \to \mathbb{R}$. This partition function $\mathcal{Z}_{LQFT}$ is precisely (up to the insertion points) our partition function $\Pi_{\gamma,\mu,\mu_\partial}^{(z_i,\alpha_i),(s_j,\beta_j)}(\hat{g}_\tau, 1)$. In the same way it is possible to write the matter term as a partition function $\mathcal{Z}_{Matter}(\hat{g}_\tau)$ and to write the formal Jacobian J as what physicists call the ghost partition function $\mathcal{Z}_{Ghost}(\hat{g}_\tau)$. The black box that we use is the following expression for $\mathcal{Z}_{LQG}$:

Hypothesis 1. *The formal ill-defined path integral $\int_{\mathcal{M}} Dg e^{-S_{EH}(g)-S_{Matter}(g)}$ can be expressed in the following way:*

$$\mathcal{Z}_{LQG} = \int D\tau \mathcal{Z}_{Ghost}(\hat{g}_\tau)\mathcal{Z}_{Matter}(\hat{g}_\tau)\mathcal{Z}_{LQFT}(\hat{g}_\tau). \tag{4.5.5}$$

The integral $\int D\tau$ is over the moduli space of the surface M. Furthermore the parameters entering in the definitions of $\mathcal{Z}_{Matter}$ and $\mathcal{Z}_{LQFT}$ must be chosen in such a way that the value of $\mathcal{Z}_{LQG}$ is independent of the choice of reference metrics $\hat{g}_\tau$ (see below).

Making rigorous sense of this statement is an open problem, see for example [19], [54]. With this new formulation we are now in the formalism of conformal field theory (CFT). Each of the partition functions $\mathcal{Z}_{Ghost}(g)$, $\mathcal{Z}_{Matter}(g)$ and $\mathcal{Z}_{LQFT}(g)$ (and also $\mathcal{Z}_{GFF}(g)$ of (4.2.10)) is expected to obey the following rule when we rescale the metric g by some Weyl factor e^σ:

$$\mathcal{Z}_{CFT}(e^\sigma g) = e^{\frac{c}{96\pi}\int_M |\partial^g\sigma|^2 d\lambda_g + 2\int_M R_g\sigma d\lambda_g}\mathcal{Z}_{CFT}(g). \tag{4.5.6}$$

This formula holds for a surface M with no boundary, see [36]. In the case of a surface with boundary Ω like our annulus, there is an extra boundary term:

$$\mathcal{Z}_{CFT}(e^\sigma g) = e^{\frac{c}{96\pi}\int_\Omega |\partial^g\sigma|^2 d\lambda_g + 2\int_\Omega R_g\sigma d\lambda_g + 4\int_{\partial\Omega} K_g\sigma d\lambda_{\partial g}}\mathcal{Z}_{CFT}(g). \tag{4.5.7}$$

The constant c is the central charge of the conformal field theory. We have $c_{GFF} = 1$, $c_{LQFT} = 1+6Q^2$ and $c_{Ghost} = -26$. The central charge of the matter c_m can vary depending on what matter fields we choose. These formulas have an important consequence: since the whole partition function $\mathcal{Z}_{LQG}$ must be independent of our choice of the reference metrics $\hat{g}_\tau$, we ask $\mathcal{Z}_{LQG}$ to be invariant by Weyl rescaling (namely if we replace $\hat{g}_\tau$ by $e^\sigma \hat{g}_\tau$ for some e^σ). This imposes the relation:

$$c_m + c_{LQFT} + c_{Ghost} = 0 \Rightarrow c_m - 25 + 6Q^2 = 0. \tag{4.5.8}$$

This gives a relation between c_m and the parameter $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$. Finally let us discuss the choice for $\mathcal{Z}_{Matter}$. The most common is to choose a power of the free field partition function

$$\mathcal{Z}_{Matter} = (\mathcal{Z}_{GFF})^{c_m} \tag{4.5.9}$$

although other choices are also possible.

4.5.2 Computation of $\mathcal{Z}_{GFF}$ and $\mathcal{Z}_{Ghost}$ for the annulus

We now explain how to obtain the expressions of $\mathcal{Z}_{GFF}$ and $\mathcal{Z}_{Ghost}$ used in section 4. It turns out it is more convenient to compute these quantities if we represent our domain by a cylinder of unit circumference and of length l. In the complex plane our cylinder is a rectangle of sides of length 1 and l where we identify both sides of length l. We will then use the map $z \mapsto e^{-2\pi i z}$ which maps this cylinder to our annulus with radii 1 and τ given that we have the relation $\tau = e^{2\pi l}$. We will first compute $\mathcal{Z}_{GFF}$ on the cylinder, the eigenvalues of $-\Delta$ are given by

$$\lambda_{m,n} = (2\pi m)^2 + (\frac{\pi n}{l})^2 \tag{4.5.10}$$

with $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ for Neumann boundary conditions. The expression of $\mathcal{Z}_{GFF}$ is given in terms of the formal product $\det{}'\Delta = \prod_{m,n}' \lambda_{m,n} = +\infty$ where the prime indicates that we omit the zero eigenvalue. The standard technique to give a finite value to this quantity is to use the zeta function regularization method. Similar computations can found in [78], [84] and for a more general discussion of the method one can consult [12]. We introduce:

$$\zeta(s) = \sum_{m,n}{}' \frac{1}{\lambda_{m,n}^s}. \tag{4.5.11}$$

This function is well defined for $\Re(s)$ sufficiently large. We can then perform an analytic continuation to the complex plane and use the following relation:

$$\det{}'\Delta = \exp \sum_{m,n}{}' \ln \lambda_{m,n} = \exp -\frac{\partial}{\partial s}\zeta(s)|_{s=0}. \tag{4.5.12}$$

With this method we obtain

$$\det{}'\Delta = 2l\eta(e^{2\pi l})^2 \tag{4.5.13}$$

with

$$\eta(x) = x^{-\frac{1}{12}} \prod_{n=1}^{\infty}(1 - x^{-2n}). \tag{4.5.14}$$

Let $A = l$ be the area of our cylinder. The partition function of the GFF is then given by [41]:

$$\mathcal{Z}_{GFF}(l, dx^2) = \sqrt{A}\det{}'\Delta^{-1/2} = \frac{1}{\sqrt{2}}\frac{1}{\eta(e^{2\pi l})}. \tag{4.5.15}$$

We must now find the link between $\mathcal{Z}_{GFF}(l, dx^2)$ and $\mathcal{Z}_{GFF}(\tau, dx^2)$. For this we will use the conformal anomaly formula (4.5.7). The cylinder equipped with the flat metric corresponds to the annulus with the metric $e^{\varphi}dx^2$ with $\varphi = -2\ln(2\pi|z|)$. To see this we introduce the coordinates $z = x+iy$ on the cylinder and the coordinates $z' = re^{i\theta}$ on the annulus. We have the relations $r = e^{2\pi y}$, $\theta = -2\pi x$ and the relation between the metrics $dz'^2 = |\psi'(z)|^2 dz^2$ with $|\psi'(z)|^2 = 4\pi^2|e^{-2\pi iz}|^2 = 4\pi^2 e^{4\pi y} = 4\pi^2 r^2$. Therefore the cylinder with the flat metric corresponds to the annulus with the metric $\frac{1}{4\pi^2 r^2}dz'^2$. To use the conformal anomaly formula we compute for $\varphi(z) = -2\ln(2\pi|z|)$,

$$\int_{\Omega}|\partial\varphi|^2 d\lambda = \int_{\theta=0}^{2\pi}\int_{r=1}^{\tau}\frac{4}{r^2}rdrd\theta = 8\pi\ln\tau,$$
$$\int_{\partial\Omega}4K\varphi d\lambda_{\partial} = -16\pi\ln\tau.$$

Therefore we get:

$$\begin{aligned}\mathcal{Z}_{GFF}(l, dx^2) = \mathcal{Z}_{GFF}(\tau, e^{\varphi}dx^2) &= \exp\left(\frac{1}{96\pi}\int_{\Omega}|\partial\varphi|^2 d\lambda + \frac{1}{96\pi}\int_{\partial\Omega}4K\varphi d\lambda_{\partial}\right)\mathcal{Z}_{GFF}(\tau, dx^2)\\ &= \tau^{-1/12}\mathcal{Z}_{GFF}(\tau, dx^2)\end{aligned}$$

which gives

$$\mathcal{Z}_{GFF}(\tau, dx^2) = \tau^{1/12}\frac{1}{\eta(\tau)}. \tag{4.5.16}$$

We have drop the irrelevant numerical factor $\frac{1}{\sqrt{2}}$. Now moving to the ghost partition function, we will start from the known result given in [52]:

$$\mathcal{Z}_{Ghost}(l, dx^2) = |\eta(e^{2\pi l})|^2. \tag{4.5.17}$$

The ghost partition function has central charge -26 so the conformal anomaly gives a factor $\tau^{-\frac{13}{6}}$. In this case we also have to include a factor coming from the change of variable on the integration measure, namely:

$$Dl = \frac{1}{2\pi\tau}D\tau. \tag{4.5.18}$$

Putting everything together we arrive at:

$$\mathcal{Z}_{Ghost}(\tau, dx^2) = \frac{1}{\tau}\tau^{-\frac{13}{6}}|\eta(\tau)|^2. \tag{4.5.19}$$

4.5.3 Green's function on the annulus

In this section we give the detailed computation of the Green's function. Similar computations can be found in [55]. We will work on the annulus of radii a and b with $a < b$. The case of interest to us is then obtained by choosing $a = 1$ and $b = \tau$. We want to find the Green's function for the following problem

$$\begin{cases} \Delta u(z) = -2\pi f(z) & \text{for} \quad z \in \Omega \\ \frac{du(r,\theta)}{dr}|_{r=a} = \alpha \int_\Omega f(r,\theta) r dr d\theta & \text{for} \quad z \in \partial\Omega_a \\ \frac{du(r,\theta)}{dr}|_{r=b} = \beta \int_\Omega f(r,\theta) r dr d\theta & \text{for} \quad z \in \partial\Omega_b \\ \int_{\partial\Omega} u d\lambda_\partial = 0 \end{cases}$$

meaning that we want to have for all functions f

$$u(r,\theta) = \int_\Omega G(r,\theta,\rho,\phi) f(\rho,\phi) \rho d\rho d\phi.$$

Here α and β are two real constants, we will discuss their values latter on. We work in polar coordinates $z = re^{i\theta}$ and write the Fourier decompositions of u, f, and G:

$$u(r,\theta) = \frac{u_0(r)}{2} + \sum_{n=1}^{\infty} (u_n^c(r)\cos(n\theta) + u_n^s(r)\sin(n\theta))$$
$$f(r,\theta) = \frac{f_0(r)}{2} + \sum_{n=1}^{\infty} (f_n^c(r)\cos(n\theta) + f_n^s(r)\sin(n\theta))$$
$$G(r,\theta,\rho,\phi) = g_0(r,\rho) + 2\sum_{n=1}^{\infty} \tilde{g}_n(r,\rho)\cos n(\theta - \phi)$$

These three functions are then linked by the relations:

$$u_0(r) = 2\pi \int g_0(r,\rho) f_0(\rho) \rho d\rho$$
$$u_n^c(r) = 2\pi \int \tilde{g}_n(r,\rho) f_n^c(\rho) \rho d\rho$$
$$u_n^s(r) = 2\pi \int \tilde{g}_n(r,\rho) f_n^s(\rho) \rho d\rho$$

The Laplace equation reads for both the sine and cosine parts of u_n and f_n and for $n \in \mathbb{N}$:

$$\frac{d}{dr}(r\frac{du_n(r)}{dr}) - \frac{n^2}{r} u_n(r) = -2\pi r f_n(r).$$

The general solutions of this equation read

$$u_0(r) = 2\pi \int_a^r \ln(x/r) f_0(x) x dx + C_0 \ln(r) + D_0$$
$$u_n(r) = \frac{\pi}{n} \int_a^r ((\frac{x}{r})^n - (\frac{r}{x})^n) f_n(x) x dx + C_n r^n + D_n r^{-n}, n \neq 0$$

where C_n and D_n are constants that will be fixed by the boundary conditions. For $n \geq 1$, our boundary conditions require $\frac{du_n(r)}{dr}$ to vanish on both boundaries. We compute:

$$\frac{du_n(r)}{dr} = -\frac{\pi}{r} \int_a^r ((\frac{x}{r})^n + (\frac{r}{x})^n) f_n(x) x dx + C_n n r^{n-1} - D_n n r^{-n-1}.$$

The constants C_n and D_n are then determined for $n > 0$ by the equations:

$$C_n n a^{n-1} - D_n n a^{-n-1} = 0$$
$$\frac{\pi}{b}\int_a^b ((\frac{x}{b})^n + (\frac{b}{x})^n) f_n(x) x dx - C_n n b^{n-1} + D_n n b^{-n-1} = 0$$

We get

$$C_n = \frac{\pi b^n}{n(b^{2n} - a^{2n})}\int_a^b ((\frac{x}{b})^n + (\frac{b}{x})^n) f_n(x) x dx$$
$$D_n = \frac{\pi b^n a^{2n}}{n(b^{2n} - a^{2n})}\int_a^b ((\frac{x}{b})^n + (\frac{b}{x})^n) f_n(x) x dx$$

and the expression of u_n

$$u_n(r) = \frac{\pi}{n}\int_a^r ((\frac{x}{r})^n - (\frac{r}{x})^n) f_n(x) x dx + \frac{\pi b^n r^n}{n(b^{2n} - a^{2n})}\int_a^b ((\frac{x}{b})^n + (\frac{b}{x})^n) f_n(x) x dx + \frac{\pi b^n a^{2n} r^{-n}}{n(b^{2n} - a^{2n})}\int_a^b ((\frac{x}{b})^n + (\frac{b}{x})^n) f_n(x) x dx.$$

This leads to a contribution to the Green's function given by:

$$\tilde{g}_n(r,\rho) = \frac{r^{-n}\rho^{-n}}{2n(b^{2n} - a^{2n})}\begin{cases} (b^{2n} + \rho^{2n})(r^{2n} + a^{2n}), & \text{for} \quad r \le \rho \\ (b^{2n} + r^{2n})(\rho^{2n} + a^{2n}), & \text{for} \quad r \ge \rho \end{cases}$$

We will rewrite the Green's function to have one uniformly convergent series and one term corresponding to the divergence when $z = z'$. In order to perform this we write for $r < \rho$:

$$\tilde{g}_n(r,\rho) = \frac{r^{-n}\rho^{-n}}{2n(b^{2n} - a^{2n})}(b^{2n} + \rho^{2n})(r^{2n} + a^{2n}) = \frac{r^{-n}\rho^{-n}}{2n}(\frac{a^{2n}}{b^{2n}(b^{2n} - a^{2n})} + \frac{1}{b^{2n}})(b^{2n} + \rho^{2n})(r^{2n} + a^{2n})$$

We will introduce:

$$g_n(r,\rho) = \frac{r^{-n}\rho^{-n}a^{2n}}{2nb^{2n}(b^{2n} - a^{2n})}\begin{cases} (b^{2n} + \rho^{2n})(r^{2n} + a^{2n}), & \text{for} \quad r \le \rho \\ (b^{2n} + r^{2n})(\rho^{2n} + a^{2n}), & \text{for} \quad r \ge \rho \end{cases}$$

We see that the Green's function can be written in terms of an absolutely convergent sum of g_n plus a term we can compute explicitly using the following formula:

$$\sum_{n=1}^{\infty}\frac{x^n}{n}\cos n\theta = -\frac{1}{2}\ln(1 - 2x\cos\theta + x^2).$$

The additional term is given by:

$$\sum_{n=1}^{\infty}\frac{1}{nb^{2n}r^n\rho^n}(b^{2n} + \rho^{2n})(r^{2n} + a^{2n})\cos n(\theta - \phi) = \ln\frac{|b^4 z^2 z'^2|}{|a^2 - z\overline{z'}||b^2 - z\overline{z'}||z - z'||b^2 z - a^2 z'|}.$$

For $r > \rho$, the expression is obtained by exchanging z and z'. This just changes the last term in the ln from $|b^2 z - a^2 z'|$ to $|a^2 z - b^2 z'|$.

Now looking at the condition for $n = 0$, we have

$$\frac{du_0(r)}{dr} = -\frac{2\pi}{r}\int_a^r f_0(x)xdx + \frac{C_0}{r}$$

and the boundary conditions give:

$$\frac{C_0}{a} = 2\pi\alpha\int_a^b f_0(x)xdx$$
$$\frac{C_0}{b} - \frac{2\pi}{b}\int_a^b f_0(x)xdx = 2\pi\beta\int_a^b f_0(x)xdx$$

Therefore, we see that for there to be a solution, we cannot pick the constants α and β arbitrarily: they are linked by the relation $\alpha a - \beta b = 1$. We can check that this relation is also a consequence of the Green-Riemann formula. The last boundary condition leads to $2\pi a u_0(a) + 2\pi b u_0(b) = 0$ which gives:

$$C_0(a\ln a + b\ln b) + D_0(a+b) + 2\pi b\int_a^b \ln(x/b)f_0(x)xdx = 0.$$

Therefore:

$$D_0 = -\frac{a\ln a + b\ln b}{a+b}C_0 - \frac{2\pi b}{a+b}\int_a^b \ln(x/b)f_0(x)xdx.$$

We have one constant left, C_0, we will choose its value to make the Green's function symmetric. The relations $C_0 = 2\pi a\alpha\int f_0(x)xdx$ and $\alpha a - \beta b = 1$ then completely determine all the constants in the problem. We have for $r \leq \rho$

$$g_0(r,\rho) = \alpha a(\frac{a\ln r/a + b\ln r/b}{a+b}) - \frac{b}{a+b}\ln\rho/b$$

and for $r \geq \rho$

$$g_0(r,\rho) = \ln\rho/r + \alpha a(\frac{a\ln r/a + b\ln r/b}{a+b}) - \frac{b}{a+b}\ln\rho/b.$$

The value of α that makes the Green's function symmetric is $\alpha = \frac{1}{a+b}$. We then have the following expression for g_0:

$$g_0(r,\rho) = \begin{cases} \frac{a^2\ln(r/a)+b^2\ln(b/\rho)+ab\ln(r/\rho)}{(a+b)^2}, & \text{for} \quad r \leq \rho \\ \frac{a^2\ln(\rho/a)+b^2\ln(b/r)+ab\ln(\rho/r)}{(a+b)^2}, & \text{for} \quad r \geq \rho \end{cases}$$

Therefore we get the final expression for our Green's function

$$G(r,\theta,\rho,\phi) = g_0(r,\rho) + 2\sum_{n=1}^{\infty} g_n(r,\rho)\cos n(\theta-\phi) + \ln\frac{|b^4z^2z'^2|}{|a^2 - z\overline{z'}||b^2 - z\overline{z'}||z-z'||b^2z - a^2z'|}$$

where again the factor $|b^2z - a^2z'|$ holds for $r < \rho$ and is replaced by $|a^2z - b^2z'|$ for $r > \rho$.

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Résumé

Cette thèse de doctorat porte sur l'étude de deux objets probabilistes, les mesures de chaos multiplicatif gaussien (GMC) et la théorie conforme des champs de Liouville (LCFT). Le GMC fut introduit par Kahane en 1985 et il s'agit aujourd'hui d'un objet extrêmement important en théorie des probabilités et en physique mathématique. Très récemment le GMC a été utilisé pour définir les fonctions de corrélation de la LCFT, une théorie qui est apparue pour la première fois en 1981 dans le célèbre article de Polyakov, "Quantum geometry of bosonic strings".

Grâce à ce lien établi entre GMC et LCFT, nous pouvons traduire les techniques de la théorie conforme des champs dans un langage probabiliste pour effectuer des calculs exacts sur les mesures de GMC. Ceci est précisément ce que nous développerons pour le GMC sur le cercle unité. Nous écrirons les équations BPZ qui fournissent des relations non triviales sur le GMC. Le résultat final est la densité de probabilité pour la masse totale de la mesure de GMC sur cercle unité ce qui résout une conjecture établie par Fyodorov et Bouchaud en 2008. Par ailleurs, il s'avère que des techniques similaires permettent également de traiter un autre cas, celui du GMC sur le segment unité, et nous obtiendrons de même des formules qui avaient été conjecturées indépendamment par Ostrovsky et par Fyodorov, Le Doussal, et Rosso en 2009.

La dernière partie de cette thèse consiste en la construction de la LCFT sur un domaine possédant la topologie d'une couronne. Nous suivrons les méthodes introduites par David-Kupiainen-Rhodes-Vargas même si de nouvelles techniques seront requises car la couronne possède deux bords et un espace des modules non trivial. Nous donnerons également des preuves plus concises de certains résultats connus.

Mots Clés

Chaos multiplicatif gaussien, Gravité quantique de Liouville, Théorie conforme des champs.

Abstract

Throughout this PhD thesis we will study two probabilistic objects, Gaussian multiplicative chaos (GMC) measures and Liouville conformal field theory (LCFT). GMC measures were first introduced by Kahane in 1985 and have grown into an extremely important field of probability theory and mathematical physics. Very recently GMC has been used to give a probabilistic definition of the correlation functions of LCFT, a theory that first appeared in Polyakov's 1981 seminal work, "Quantum geometry of bosonic strings".

Once the connection between GMC and LCFT is established, one can hope to translate the techniques of conformal field theory in a probabilistic framework to perform exact computations on the GMC measures. This is precisely what we develop for GMC on the unit circle. We write down the BPZ equations which lead to non-trivial relations on the GMC. Our final result is an exact probability density for the total mass of the GMC measure on the unit circle. This proves a conjecture of Fyodorov and Bouchaud stated in 2008. Furthermore, it turns out that the same techniques also work on a more difficult model, the GMC on the unit interval, and thus we also prove conjectures put forward independently by Ostrovsky and by Fyodorov, Le Doussal, and Rosso in 2009.

The last part of this thesis deals with the construction of LCFT on a domain with the topology of an annulus. We follow the techniques introduced by David-Kupiainen-Rhodes-Vargas although novel ingredients are required as the annulus possesses two boundaries and a non-trivial moduli space. We also provide more direct proofs of known results.

Keywords

Gaussian multiplicative chaos, Liouville quantum gravity, Conformal field theory.