ANALYTICITY AND SYMMETRY OF VIRASORO CONFORMAL BLOCKS VIA LIOUVILLE CFT

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ABSTRACT. Virasoro conformal blocks are a family of functions that encode the universal structure for 2D conformal field theory (CFT) in the conformal bootstrap framework. Two of the most basic conformal blocks are the one for the four-point sphere and for the one-point torus cases. They are defined as power series of the complex parameter (i.e. the modular parameter) encoding the corresponding marked Riemann surfaces, which are determined by the Virasoro algebra indexed by a central charge c. Moreover, each marked point carries a parameter called the external momentum and there is an additional parameter called the internal momentum. It is a longstanding conjecture that the these two power series have convergence radius 1. Moreover, they are meromorphic in their internal momentum where the location of the poles are dictated by the Kac table. In this paper we prove this conjecture when the central charge $c > 25$ and the external moments belongs to a certain range. Moreover, we prove Ponsot and Teschner's formula for the fusion kernel of the four-point spherical blocks and Teschner's formula for the modular kernel of the one-point torus block. These kernel encode the symmetry of the space of conformal blocks with respect to the mapping class group. Our proof is based on the recently developed conformal bootstrap for Liouville CFT constructed using the Gaussian multiplicative chaos (GMC). As a byproduct, we derived GMC expression of the four-point sphere and one-point torus conformal blocks.

CONTENTS

1. INTRODUCTION

A conformal field theory (CFT) is a way to construct random functions on Riemannian manifolds that transform covariantly under conformal (i.e. angle preserving) mappings. Since the seminal work of Belavin-Polyakov-Zamolodchikov [BPZ84], two dimensional (2D) CFT has grown into one of the most prominent branches of theoretical physics, with applications to 2D statistical physics and string theory, as well as far reaching consequences in mathematics; see e.g. [DFMS97]. The paper [BPZ84] introduced a schematic program called the *conformal bootstrap* to exactly solve correlation functions of a given 2D CFT in terms of its 3-point sphere correlation functions and certain power series called **conformal blocks**. These conformal blocks are completely specified by the Virasoro algebra that encodes the infinitesimal local conformal symmetries, and they only depend on the specific CFT through a single parameter called the *central charge*. Outside of CFT, conformal blocks are related to Nekrasov partition functions in gauge theory via the Alday-Gaiotto-Tachikawa correspondence [AGT10], solutions to Painlevé-type equations [GIL12], and quantum Teichmüller theory and representation of quantum groups [PT99, PT01, TV15], among other things.

In this paper, we prove analyticity and symmetry properties for these conformal blocks in the two most fundamental cases: the one of the one-point torus and of the four-point sphere. Our proof strategy requires

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the probabilistic construction of the **Liouville conformal field theory** (LCFT) giving rigorous meaning to the path integral formalism of quantum field theory on the sphere in [DKRV16] and on other surfaces in [DRV16, HRV18, GRV19]. The construction is via **Gaussian multiplicative chaos** (GMC), a random measure defined by exponentiating the Gaussian free field (see e.g. [RV14, Ber17]). LCFT depends on a coupling constant $\gamma \in (0,2)$ which is in bijection with the central charge c via

$$c = 1 + 6Q^2 \in (25, \infty), \qquad \text{where } Q = \frac{\gamma}{2} + \frac{2}{\gamma}. \tag{1.1}$$

The conformal bootstrap for Liouville CFT has recently been proved in this probabilistic framework, first in the case of the Riemann sphere [GKRV20], then for general boundaryless Riemann surfaces [GKRV21], and lastly in the case of the annulus in [Wu22]. We will take these boundary bootstrap statements as the starting point of our proofs. We now move to defining the conformal blocks of interest purely in terms of the Virasoro algebra and stating the main analyticity result for these functions.

1.1. **Analyticity of the 4-point spherical and 1-point toric conformal blocks.** We start by recalling the definition of conformal blocks as power series defined in terms of the Virasoro algebra. The Virasoro algebra Vir with central charge c is the associative algebra with generators $\{L_n\}_{n\in\mathbb{Z}}$ and $\mathbf{1}$ and relations

$$\mathbf{1}\mathbf{L}_n = \mathbf{L}_n\mathbf{1} \quad \text{and} \quad [\mathbf{L}_n, \mathbf{L}_m] = (n-m)\mathbf{L}_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n,-m}\mathbf{1} \quad \text{for } m, n \in \mathbb{Z}. \tag{1.2}$$

A sequence of integers $\nu = (\nu_i)_{i\geq 1}$ is called a Young diagram if the mapping $i \mapsto \nu_i$ is non-increasing and if $\nu_i = 0$ for i sufficiently large. We denote by $|\nu| := \sum_{i\geq 1} \nu_i$ the length of the Young diagram. Given a Young diagram ν we denote

$$\mathbf{L}_{-\nu} = \mathbf{L}_{-\nu_k} \cdots \mathbf{L}_{-\nu_1} \quad \text{and} \quad \mathbf{L}_{\nu} = \mathbf{L}_{\nu_1} \cdots \mathbf{L}_{\nu_k}.$$

For Young diagrams ν and ν' satisfying $|\nu| = |\nu'|$, one can check that (see e.g. Lemma A.2 in [GKRV20])

$$\mathbf{L}_{\nu}\mathbf{L}_{-\nu'}\mathbf{1} = \sum_{k\geq 0} a_k(\nu,\nu')\mathbf{L}_0^k\mathbf{1}. \tag{1.3}$$

where the coefficients $a_k(\nu,\nu')$ are purely determined by the Virasoro algebra. We yields the dependence of $a_k(\nu,\nu')$ on c as we fix the central charge as a globally parameter. The *Schapovalov form* is then defined as

$$F_{\Delta}(\nu,\nu') = \sum_{k\geq 0} a_k(\nu,\nu')\Delta^k. \tag{1.4}$$

For $N \geq 0$, let $F_{\Delta,N}(\nu,\nu') = (F_\Delta(\nu,\nu'))_{|\nu|=|\nu'|=N}$. By Kac determinant formula (see [FF83]), the matrix $F_{\Delta,N}$ is invertible if and only if $\Delta \in \mathbb{C}$ is not in the set

$$\mathrm{DW}_c(N) := \{\frac{c-1}{24} - (rb + sb^{-1})^2 : r, s \text{ are positive integers such that } rs \leq N\},$$

where b satisfies $c = 1 + 6(b + b^{-1})^2$. Although b has two choices the set $\mathrm{DW}_c(N)$ is independent of this choice. The set of degenerate weights of the Virasoro algebra is given by

$$\mathrm{DW}_c := \cup_{N\geq 0}\mathrm{DW}_c(N) = \{\Delta_{r,s} = \frac{c-1}{24} - (rb + sb^{-1})^2 : r, s \text{ are positive integers}\}. \tag{1.5}$$

Fix c and as $\alpha_1, \alpha_2, \alpha_3, \alpha_4, P$ as parameters. Then define the following quantities:

$$\Delta_{\alpha_i} = \frac{\alpha_i}{2}(Q - \frac{\alpha_i}{2}), \quad \Delta_{Q+\mathbf{i}P} = \frac{Q^2}{4} + \frac{P^2}{4} = \frac{c-1}{24} + \frac{P^2}{4}.$$

For integers $n \geq 0$ and $\Delta_P \notin \mathrm{DW}_c$, let

$$\beta_n(\Delta_{Q+\mathbf{i}P}; \Delta_{\alpha_1}, \Delta_{\alpha_2}, \Delta_{\alpha_3}, \Delta_{\alpha_4}) := \sum_{|\nu|,|\nu'|=n} v(\Delta_{\alpha_1}, \Delta_{\alpha_2}, \Delta_{Q+\mathbf{i}P}, \nu) F^{-1}_{\Delta_{Q+\mathbf{i}P}}(\nu,\nu') v(\Delta_{\alpha_4}, \Delta_{\alpha_3}, \Delta_{Q+\mathbf{i}P}, \nu'), \tag{1.6}$$

where the sum is over Young diagrams and $v(\cdot,\cdot,\cdot,\cdot)$ is given by

$$v(\Delta, \Delta', \Delta'', \nu) := \prod_{j=1}^{k} (\nu_j\Delta' - \Delta + \Delta'' + \sum_{u<j} \nu_u). \tag{1.7}$$

Definition 1.1. The s-channel four-point spherical conformal block is defined as the formal power series

$$\mathcal{F}^{c,\text{sphere}}_{\Delta_{\alpha_1},\Delta_{\alpha_2},\Delta_{\alpha_3},\Delta_{\alpha_4}}(z;\Delta_{Q+\mathbf{i}P}) = \sum_{n=0}^{\infty} \beta_n(\Delta_{Q+\mathbf{i}P};\Delta_{\alpha_1},\Delta_{\alpha_2},\Delta_{\alpha_3},\Delta_{\alpha_4}) z^n. \tag{1.8}$$

Remark. In some literature the s-channel block is defined as $z^{\Delta_{Q+\mathbf{i}P}-\Delta_{\alpha_1}-\Delta_{\alpha_2}}\mathcal{F}^{c,\text{sphere}}_{\Delta_{\alpha_1},\Delta_{\alpha_2},\Delta_{\alpha_3},\Delta_{\alpha_4}}(z;\Delta_{Q+\mathbf{i}P})$ instead.

The first main result of our paper is to the realization of the four-point spherical conformal block as a function holomorphic in z in the unit disk, meromorphic in $\Delta_{Q+\mathbf{i}P}$ with poles at the degenerate weights, for $c > 25$ and $(\Delta_{\alpha_1},\Delta_{\alpha_2},\Delta_{\alpha_3},\Delta_{\alpha_4})$ in some appropriate range. In our statement below for each $\Delta \in \mathrm{DW}_c$, we let $m(\Delta)$ be the cardinality of the set $\{\text{integers } r,s \geq 1 : \frac{c-1}{24} - (rb+sb^{-1})^2 = \Delta\}$ and called it the multiplicity of Δ in DW_c. When c is such that b^2 is irrational, then each $\Delta \in \mathrm{DW}_c$ has multiplicity one. Otherwise there exists degenerates weights with finite multiplicities greater than one.

Theorem 1.2. Fix $c > 25$ and $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ satisfying $\alpha_i < Q$, $\alpha_1+\alpha_2 > Q$, $\alpha_3+\alpha_4 > Q$, $\alpha_1+\alpha_4 > Q$, $\alpha_2+\alpha_3 > Q$. There exists a function $\mathcal{F}^{\text{sph}}(z;\Delta)$ for complex (z,Δ) (depending on $c,\Delta_{\alpha_1},\Delta_{\alpha_2},\Delta_{\alpha_3},\Delta_{\alpha_4}$) with the following properties:

(a). $\mathcal{F}^{\text{sph}}(z;\Delta)$ is analytic in (z,Δ) for $|z|<1$ and $\Delta \notin \mathrm{DW}_c$;
(b). For $|z_0|<1$ and $\Delta_{r,s} \in \mathrm{DW}_c$, $(\Delta-\Delta_{r,s})^{m(\Delta_{r,s})}\mathcal{F}^{\text{sph}}(z;\Delta)$ is analytic in a neighborhood of $(z_0,\Delta_{r,s})$;
(c). $\frac{\partial^n}{\partial z^n}\mathcal{F}^{\text{sph}}(0;\Delta) = n!\beta_n(\Delta;\Delta_{\alpha_1},\Delta_{\alpha_2},\Delta_{\alpha_3},\Delta_{\alpha_4})$ for $\Delta \notin \mathrm{DW}_c$ with β_n in (1.8);
(d). For $\Delta_{r,s} \in \mathrm{DW}_c$, one has:

$$\operatorname*{Res}_{\Delta=\Delta_{m,n}} \mathcal{F}^{\text{sph}}(z;\Delta) = A_{m,n} z^{mn}\mathcal{F}^{\text{sph}}(z;\Delta_{-m,n}).$$

We now turn to the definition of the 1-point toric conformal block. For $\Delta \in \mathbb{C}$, the *Verma module* M_Δ of *highest weight* Δ is a Vir-module defined as follows. There exists a vector $v_\Delta \in M_\Delta$, called a *highest weight* vector, such that

$$\mathbf{L}_n v_\Delta = 0 \text{ for } n>0 \qquad \text{and} \qquad \mathbf{L}_0 v_\Delta = \Delta v_\Delta. \tag{1.9}$$

Moreover, the set of vectors $\{\mathbf{L}_{-\nu}v_\Delta\}$ indexed by Young diagrams form a basis of M_Δ. The action of Vir on M_Δ is given by commuting the action of a generator $\mathbf{L}_n$ on a basis vector to create a linear combination of other basis vectors using the relations (1.2). Given Δ_1,Δ_2,Δ, the operator product expansion in CFT leads to a bilinear form $(\cdot \mid \cdot)_\Delta$ on $M_{\Delta_1}\times M_{\Delta_2}$ which can be uniquely specified by

$$\begin{aligned}(\mathbf{L}_{-\nu}v_{\Delta_1} \mid \mathbf{L}_{-n}\mathbf{L}_{-\tilde\nu}v_{\Delta_2})_\Delta &= (\mathbf{L}_n\mathbf{L}_{-\nu}v_{\Delta_1} \mid \mathbf{L}_{-\tilde\nu}v_{\Delta_2})_\Delta + (-\Delta+|\nu|-|\tilde\nu|+(n+1)\Delta)(\mathbf{L}_{-\nu}v_{\Delta_1} \mid \mathbf{L}_{-\tilde\nu}v_{\Delta_2})_\Delta;\\ (v_{\Delta_1} \mid v_{\Delta_2})_\Delta &= 1.\end{aligned} \tag{1.10}$$

Now we set $w(\Delta,\Delta_1,\Delta_2,\nu,\tilde\nu) = (\mathbf{L}_{-\nu}v_{\Delta_1} \mid \mathbf{L}_{-\tilde\nu}v_{\Delta_2})_\Delta$ and

$$W_N(\Delta,\Delta_1,\Delta_2) = (w(\Delta,\Delta_1,\Delta_2,\nu,\tilde\nu))_{|\nu|=|\tilde\nu|=N} \quad \text{for } N \geq 0. \tag{1.11}$$

For fixed Young diagrams $\nu,\tilde\nu$, one can check that $w(\Delta,\Delta_1,\Delta_2,\nu,\tilde\nu)$ is a polynomial in Δ,Δ_1,Δ_2. Moreover, similar to the Shapovalov matrix $F_{\Delta,N}$, for fixed $(\Delta,\Delta_1,\Delta_2)$, we can view $W_N(\Delta,\Delta_1,\Delta_2)$ as a matrix whose entries are indexed by Young diagrams of length N.

Definition 1.3. The one-point toric conformal block is defined as the formal power series

$$\mathcal{F}^{c,\text{torus}}_{\Delta_{\alpha_1}}(q;\Delta_{Q+\mathbf{i}P}) = \sum_{n=0}^{\infty} \mathrm{Tr}(F^{-1}_{\Delta_{Q+\mathbf{i}P},n}\cdot W_n(\Delta_{\alpha_1},\Delta_{Q+\mathbf{i}P},\Delta_{Q+\mathbf{i}P}))q^n. \tag{1.12}$$

We now state the analogue of Theorem 1.2 for the 1-point torus case.

Theorem 1.4. Fix $c > 25$ and $\alpha \in (0,Q)$. There exists a function $\mathcal{F}^{\text{tor}}(q;\Delta)$ for complex (q,Δ) (depending on c,Δ_{α_1}) with the following properties:

(a). $\mathcal{F}^{\text{tor}}(q;\Delta)$ is analytic in (q,Δ) for $|q|<1$ and $\Delta \notin \mathrm{DW}_c$;
(b). For $|q_0|<1$ and $\Delta_{r,s} \in \mathrm{DW}_c$, $(\Delta-\Delta_{r,s})^{m(\Delta_{r,s})}\mathcal{F}^{\text{tor}}(q;\Delta)$ is analytic in a neighborhood of $(q_0,\Delta_{r,s})$;
(c). $\frac{\partial^n}{\partial q^n}\mathcal{F}^{\text{tor}}(0;\Delta) = n!\mathrm{Tr}(F^{-1}_{\Delta,n}\cdot W_n(\Delta_{\alpha_1},\Delta,\Delta))$ for $\Delta \notin \mathrm{DW}_c$;

(d). Viewed as a meromorphic function in $\Delta \in \mathbb{C}$, $\mathcal{F}^{\mathrm{tor}}(q;\Delta)$ obeys:

$$\mathcal{F}^{\mathrm{tor}}(q;\Delta) = \sum_{n,m=1}^{\infty} q^{2mn} \frac{R_{\gamma,m,n}(\alpha)}{P^2 - P_{m,n}^2} \mathcal{F}^{\mathrm{tor}}(q;\Delta_{-m,n}) + q^{\frac{1}{12}} \eta(q)^{-1}.$$

1.2. **Fusion and modular symmetries.** We now give the fusion transformation for the four point sphere conformal block. It will related the expansion of the conformal block around $z=0$ and $z=1$.

Theorem 1.5. Let $z \in (0,1)$, let α_i satisfy $\alpha_i < Q$, $\alpha_1+\alpha_2 > Q$, $\alpha_3+\alpha_4 > Q$, $\alpha_1+\alpha_4 > Q$, $\alpha_2+\alpha_3 > Q$ and $P \in \mathbb{R}$. Then the following holds

$$z^{\frac{1}{2}P^2} \mathcal{F}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(z,P) = C(z) \int_{\mathbb{R}} \mathcal{M}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(P,P')(1-z)^{\frac{1}{2}(P')^2} \mathcal{F}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(1-z,P')dP', \tag{1.13}$$

where $C(z) = z^{(\Delta_{\alpha_1}+\Delta_{\alpha_2}-\frac{Q^2}{2})}(1-z)^{(\frac{Q^2}{2}-\Delta_{\alpha_3}-\Delta_{\alpha_2})}$ and where the fusion kernel is given by the formula (3.10).

We also have an analogue result for the one-point torus conformal block. Recall the modular parameter $\tau \in \mathbb{H}$. Define $q = e^{\mathbf{i}\pi\tau}$ and $\widetilde{q} = e^{-\frac{\mathbf{i}\pi}{\tau}}$. Then the following modular transformation relates the values of the one-point torus block at q and $\widetilde{q}$.

Theorem 1.6. For $\alpha \in (0,Q)$, $q \in (0,1)$, $P \in \mathbb{R}$ one has

$$q^{-\frac{1}{12}+\frac{1}{2}P^2} \mathcal{F}^{\mathsf{torus}}_{\alpha}(q,P) = \int_{\mathbb{R}} \mathcal{M}^{\mathsf{torus}}_{\alpha}(P,P') \widetilde{q}^{-\frac{1}{12}+\frac{1}{2}(P')^2} \mathcal{F}^{\mathsf{torus}}_{\alpha}(\widetilde{q},P')dP', \tag{1.14}$$

where the modular kernel $\mathcal{M}^{\mathsf{torus}}_{\alpha}(P,P')$ has expression given by (5.3).

2. Background on Liouville conformal field theory

Throughout most this section we work on $\mathbb{H}$ as our base domain, although by conformal invariance it would be equivalent to work on the unit disk $\mathbb{D}$ or on any simply connected domain. In the last subsection we repeat all the constructions on the annulus $\mathbb{A}$.

2.1. **Gaussian free field and Gaussian multiplicative chaos.** Let h be the free boundary Gaussian free field on the upper half plane $\mathbb{H}$ with covariance kernel

$$\mathbb{E}[h(x)h(y)] = \log \frac{1}{|x-y||x-\bar{y}|} + 2\log|x|_+ + 2\log|y|_+, \tag{2.1}$$

where $|x|_+ := \max(|x|,1)$ and in the sense that $\mathbb{E}[(h,f)(h,g)] = \iint f(x)\mathbb{E}[h(x)h(y)]g(y)dxdy$, for smooth test functions f and g. Let $P_{\mathbb{H}}$ be the law of h, so that $P_{\mathbb{H}}$ is a probability measure on the negatively indexed Sobolev space $H^{-1}(\mathbb{H})$. This particular covariance kernel (2.1) corresponds to requiring the field to have average 0 on the upper-half unit circle.

Given a sample h from $P_{\mathbb{H}}$, let $h_\epsilon(z)$ denote the average of h over $\{w \in \mathbb{H} : |w-z| = \epsilon\}$. The associated quantum area and length measure - also known as Gaussian multiplicative chaos measures - are defined by:

$$\mathcal{A}_h = \lim_{\varepsilon\to 0} \epsilon^{\frac{\gamma^2}{2}} e^{\gamma h_\epsilon(z)} d^2 z, \quad \text{and} \quad \mathcal{L}_h = \lim_{\varepsilon \to 0} \epsilon^{\frac{\gamma^2}{4}} e^{\frac{\gamma}{2} h_\epsilon(z)} dz. \tag{2.2}$$

These limits hold in probability and against all suitable test functions.

2.2. **Probabilistic definition of LCFT correlations on the upper-half plane.** Before introducing the correlation functions, we introduce the Liouville field, possibly with boundary insertions, which is constructed from the Gaussian free field h.

Definition 2.1 (Liouville field)**.** Let $(h,\mathbf{c})$ be sampled from $P_{\mathbb{H}} \times [e^{-Qc}dc]$ and set $\phi = h(z) - 2Q\log|z|_+ + \mathbf{c}$. We write $\mathrm{LF}_{\mathbb{H}}$ as the law of ϕ, and call a sample from $\mathrm{LF}_{\mathbb{H}}$ a *Liouville field on* $\mathbb{H}$.

Definition 2.2 (Liouville field with insertions)**.** Let $(\alpha_j, s_j) \in \mathbb{R} \times \partial\mathbb{H}$ for $j = 1, \ldots, M$, where $M \geq 1$ and the s_j are pairwise distinct. Sample $(h,\mathbf{c})$ from $C_{\mathbb{H}}^{(\alpha_j,s_j)_j} P_{\mathbb{H}} \times [e^{(\frac{1}{2}\sum_j \alpha_j - Q)c}dc]$ where

$$C_{\mathbb{H}}^{(\alpha_j,s_j)_j} = \prod_{j=1}^{M} |s_j|_+^{-\alpha_j(Q-\frac{\alpha_j}{2})} \prod_{1\leq j<k\leq M} e^{\frac{\alpha_j\alpha_k}{4} G_{\mathbb{H}}(s_j,s_k)}.$$

Let $\phi(z) = h(z) - 2Q\log|z|_+ + \sum_{j=1}^M \frac{\alpha_j}{2} G_{\mathbb{H}}(z,s_j) + \mathbf{c}$. We write $\mathrm{LF}_{\mathbb{H}}^{(\alpha_j,s_j)_j}$ for the law of ϕ and call a sample from $\mathrm{LF}_{\mathbb{H}}^{(\alpha_j,s_j)_j}$ the *Liouville field on* $\mathbb{H}$ *with insertions* $(\alpha_j,s_j)_{1\le j\le M}$.

We can also define Liouville fields with an insertion at ∞. Fix $z\in(0,1)$. We will need the case $\mathrm{LF}_{\mathbb{H}}^{(\alpha_1,0),(\alpha_2,z),(\alpha_3,1),(\alpha_4,\infty)}$, which can be defined by $\lim_{s\to\infty}|s|^{2\Delta_4}\mathrm{LF}_{\mathbb{H}}^{(\alpha_1,0),(\alpha_2,z),(\alpha_3,1),(\alpha_4,s)}$ with $\Delta_4 = \frac{\alpha_4}{2}(Q-\frac{\alpha_4}{2})$. Here we give a more explicit definition without using a limiting procedure.

Definition 2.3. Fix $\alpha_1,\alpha_2,\alpha_3,\alpha_4\in\mathbb{R}$ and $z\in(0,1)$. Set $s_1=0, s_2=z, s_3=1, s_4=\infty$ and $G_{\mathbb{H}}(z,\infty) = 2\log|z|_+$. Sample $(h,\mathbf{c})$ from $|z|^{-\frac{\alpha_1\alpha_2}{2}}|1-z|^{-\frac{\alpha_2\alpha_3}{2}}P_{\mathbb{H}}\times[e^{(\frac{1}{2}\sum_j\alpha_j-Q)c}dc]$. We write $\mathrm{LF}_{\mathbb{H}}^{(\alpha_1,0),(\alpha_2,z),(\alpha_3,1)(\alpha_4,\infty)}$ for the law of ϕ where $\phi(z) = h(z) - 2Q\log|z|_+ + \sum_{j=1}^4\frac{\alpha_j}{2}G_{\mathbb{H}}(z,s_j) + \mathbf{c}$.

With these definitions we can now give the definition of the boundary four-point function of LCFT.

Definition 2.4. Let $\mu_i\ge 0$ for $i=1,2,3,4$ with at least one parameter strictly positive. Suppose $\alpha_1,\alpha_2,\alpha_3,\alpha_4\in\mathbb{R}$ satisfy the Seiberg bounds:

$$\sum_{i=1}^4 \alpha_i > 2Q \quad\text{and}\quad \alpha_i < Q. \tag{2.3}$$

Then using the notation $(z_1,z_2,z_3,z_4) = (0,z,1,\infty)$, the boundary four-point function is defined by:
(2.4)
$$\left\langle \prod_{i=1}^4 B_{\alpha_i}^{\mu_i,\mu_{i+1}}(z_i)\right\rangle := \int e^{-\mu_1\mathcal{L}_\phi(-\infty,0)-\mu_2\mathcal{L}_\phi(0,z)-\mu_3\mathcal{L}_\phi(z,1)-\mu_4\mathcal{L}_\phi(1,+\infty)}\ \mathrm{LF}_{\mathbb{H}}^{(\alpha_1,0),(\alpha_2,z),(\alpha_3,1),(\alpha_4,\infty)}(\mathrm{d}\phi).$$

Here the notation $B_{\alpha_i}^{\mu_i,\mu_{i+1}}(z_i)$ indicates we have a boundary insertion of weight α_i at z_i with boundary cosmological constant μ_i to the left of z_i and μ_{i+1} to the right. By convention $\mu_5=\mu_1$.

2.3. **Structure constants of boundary LCFT.** The first step in solving boundary Liouville CFT is to compute its structure constants. This has been performed in the probabilistic framework in the works [RZ20]. We recall here these results in the case where the bulk Liouville potential is set to zero. We list the analytic expressions for the bulk one-point function U, the bulk-boundary correlator G, and the boundary three-point function H. These formulas can also be defined probabilistically in an appropriate parameter range (as the boundary four-point function given above) but for our purposes we will only be working with the analytic expressions.

- **Bulk one-point function.** Consider parameters $\mu_B>0$, $\beta\in\mathbb{C}$. The bulk one-point function has analytic expression given by:

$$U_{\mu_B}(\beta) = \frac{2}{\gamma}\Gamma(\frac{2(\beta-Q)}{\gamma})\mu_B^{\frac{2(Q-\beta)}{\gamma}}\left(\frac{2^{-\frac{\gamma\beta}{2}}2\pi}{\Gamma(1-\frac{\gamma^2}{4})}\right)^{\frac{2}{\gamma}(Q-\beta)}\Gamma(\frac{\gamma\beta}{2}-\frac{\gamma^2}{4}). \tag{2.5}$$

- **Bulk-boundary correlator.** Consider parameters $\mu_B>0$, $\alpha,\beta\in\mathbb{C}$. The one-point bulk one-point boundary function has analytic expression given by:

$$G_{\mu_B}(\alpha,\beta) = \frac{2}{\gamma}\Gamma(\frac{2(\beta-Q)+\alpha}{\gamma})\mu_B^{\frac{2(Q-\beta)-\alpha}{\gamma}}\left(\frac{2^{-\frac{\gamma\beta}{2}+\frac{\gamma\alpha}{4}}2\pi}{\Gamma(1-\frac{\gamma^2}{4})}\right)^{\frac{2}{\gamma}(Q-\beta-\frac{\alpha}{2})} \times\frac{\Gamma(\frac{\gamma\beta}{2}+\frac{\gamma\alpha}{4}-\frac{\gamma^2}{4})\Gamma_{\frac{\gamma}{2}}(\beta-\frac{\alpha}{2})\Gamma_{\frac{\gamma}{2}}(\beta+\frac{\alpha}{2})\Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2})^2}{\Gamma_{\frac{\gamma}{2}}(Q-\alpha)\Gamma_{\frac{\gamma}{2}}(\beta)^2\Gamma_{\frac{\gamma}{2}}(Q)}. \tag{2.6}$$

  Note here that the parameter α corresponds to the boundary insertion and the parameter β to the bulk insertion, which is the opposite of the convention used in (cite).
- **Boundary three-point function.** Consider parameters $\beta_1,\beta_2,\beta_3\in\mathbb{C}$ and $\mu_1,\mu_2,\mu_3\in\{z\in\mathbb{C}|\mathrm{Re}(z)>0\}$. Given these μ_i we define variables σ_i by the relation $\mu_i = e^{\gamma\mathbf{i}\pi(\sigma_i-\frac{Q}{2})}$ with $\sigma_i=\frac{Q}{2}$ for $\mu_i=1$. The boundary three-point function is then given by

$$H_{(\mu_1,\mu_2,\mu_3)}^{(\beta_1,\beta_2,\beta_3)} = \frac{2}{\gamma}\Gamma(\frac{\beta_1+\beta_2+\beta_3-2Q}{\gamma})\overline{H}_{(\mu_1,\mu_2,\mu_3)}^{(\beta_1,\beta_2,\beta_3)},$$

where $\overline{H}^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)}$ is a meromorphic function on $\mathbb{C}^6$ of its six parameters β_i, σ_i, which under the condition $\mathrm{Re}\left(Q - \sigma_3 + \sigma_2 - \frac{\beta_2}{2}\right) > 0$ can be represented by the formula:

$$
\begin{aligned}
(2.7)\quad & \overline{H}^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)} \\
&= \frac{(2\pi)^{\frac{2Q-\overline{\beta}}{\gamma}+1}(\frac{2}{\gamma})^{(\frac{\gamma}{2}-\frac{2}{\gamma})(Q-\frac{\overline{\beta}}{2})-1}}{\Gamma(1-\frac{\gamma^2}{4})^{\frac{2Q-\overline{\beta}}{\gamma}}\Gamma(\frac{\overline{\beta}-2Q}{\gamma})} \frac{\Gamma_{\frac{\gamma}{2}}(2Q-\frac{\overline{\beta}}{2})\Gamma_{\frac{\gamma}{2}}(\frac{\beta_1+\beta_3-\beta_2}{2})\Gamma_{\frac{\gamma}{2}}(Q-\frac{\beta_1+\beta_2-\beta_3}{2})\Gamma_{\frac{\gamma}{2}}(Q-\frac{\beta_2+\beta_3-\beta_1}{2})}{\Gamma_{\frac{\gamma}{2}}(Q)\Gamma_{\frac{\gamma}{2}}(Q-\beta_1)\Gamma_{\frac{\gamma}{2}}(Q-\beta_2)\Gamma_{\frac{\gamma}{2}}(Q-\beta_3)} \\
&\times \frac{e^{i\frac{\pi}{2}(-(2Q-\frac{\beta_1}{2}-\sigma_1-\sigma_2)(Q-\frac{\beta_1}{2}-\sigma_1-\sigma_2)+(Q+\frac{\beta_2}{2}-\sigma_2-\sigma_3)(\frac{\beta_2}{2}-\sigma_2-\sigma_3)+(Q+\frac{\beta_3}{2}-\sigma_1-\sigma_3)(\frac{\beta_3}{2}-\sigma_1-\sigma_3)-2\sigma_3(2\sigma_3-Q))}}{S_{\frac{\gamma}{2}}(\frac{\beta_1}{2}+\sigma_1-\sigma_2)S_{\frac{\gamma}{2}}(\frac{\beta_3}{2}+\sigma_3-\sigma_1)} \\
&\times \int_{\mathcal{C}} \frac{S_{\frac{\gamma}{2}}(Q-\frac{\beta_2}{2}+\sigma_3-\sigma_2+r)S_{\frac{\gamma}{2}}(\frac{\beta_3}{2}+\sigma_3-\sigma_1+r)S_{\frac{\gamma}{2}}(Q-\frac{\beta_3}{2}+\sigma_3-\sigma_1+r)}{S_{\frac{\gamma}{2}}(Q+\frac{\beta_1}{2}-\frac{\beta_2}{2}+\sigma_3-\sigma_1+r)S_{\frac{\gamma}{2}}(2Q-\frac{\beta_1}{2}-\frac{\beta_2}{2}+\sigma_3-\sigma_1+r)S_{\frac{\gamma}{2}}(Q+r)} e^{i\pi(-\frac{\beta_2}{2}+\sigma_2-\sigma_3)r}\frac{dr}{i}.
\end{aligned}
$$

In the integral appearing above the contour $\mathcal{C}$ goes from $-i\infty$ to $i\infty$ passing to the right of the poles at $r = -(Q-\frac{\beta_2}{2}+\sigma_3-\sigma_2)-n\frac{\gamma}{2}-m\frac{2}{\gamma}$, $r = -(\frac{\beta_3}{2}+\sigma_3-\sigma_1)-n\frac{\gamma}{2}-m\frac{2}{\gamma}$, $r = -(Q-\frac{\beta_3}{2}+\sigma_3-\sigma_1)-n\frac{\gamma}{2}-m\frac{2}{\gamma}$ and to the left of the poles at $r = -(\frac{\beta_1}{2}-\frac{\beta_2}{2}+\sigma_3-\sigma_1)+n\frac{\gamma}{2}+m\frac{2}{\gamma}$, $r = -(Q-\frac{\beta_1}{2}-\frac{\beta_2}{2}+\sigma_3-\sigma_1)+n\frac{\gamma}{2}+m\frac{2}{\gamma}$, $r = n\frac{\gamma}{2}+m\frac{2}{\gamma}$ with $m,n \in \mathbb{N}$. See appendix A.2 for more details on this expression.

Note again that all these expressions correspond to having set the bulk cosmological constant of Liouville CFT to 0. We will also use the special cases of the boundary three-point function where one or two of the boundary cosmological constants are set to zero. This reduction has been performed in [RZ20]. We state it here as a lemma.

Lemma 2.5. Let $\beta_1,\beta_2,\beta_3 \in \mathbb{C}$ and $\mu_2,\mu_3 > 0$. Define as before σ_2,σ_3 by the relation $\mu_i = e^{\gamma \mathbf{i}\pi(\sigma_i - \frac{Q}{2})}$ with $\sigma_i = \frac{Q}{2}$ for $\mu_i = 1$. In the case of a single non-zero cosmological constant, one has the simple scaling

$$
H^{(\beta_1,\beta_2,\beta_3)}_{(0,\mu_2,0)} = \frac{2}{\gamma}\Gamma(\frac{\beta_1+\beta_2+\beta_3-2Q}{\gamma})\mu_2^{\frac{2Q-\beta_1-\beta_2-\beta_3}{\gamma}}\overline{H}^{(\beta_1,\beta_2,\beta_3)}_{(0,1,0)},
$$

and the expression:

$$
\overline{H}^{(\beta_1,\beta_2,\beta_3)}_{(0,1,0)} = \frac{(2\pi)^{\frac{2Q-\overline{\beta}}{\gamma}+1}(\frac{2}{\gamma})^{(\frac{\gamma}{2}-\frac{2}{\gamma})(Q-\frac{\overline{\beta}}{2})-1}}{\Gamma(1-\frac{\gamma^2}{4})^{\frac{2Q-\overline{\beta}}{\gamma}}\Gamma(\frac{\overline{\beta}-2Q}{\gamma})} \frac{\Gamma_{\frac{\gamma}{2}}(\frac{\overline{\beta}}{2}-Q)\Gamma_{\frac{\gamma}{2}}(\frac{\beta_1+\beta_3-\beta_2}{2})\Gamma_{\frac{\gamma}{2}}(\frac{\beta_2+\beta_3-\beta_1}{2})\Gamma_{\frac{\gamma}{2}}(Q-\frac{\beta_1+\beta_2-\beta_3}{2})}{\Gamma_{\frac{\gamma}{2}}(Q)\Gamma_{\frac{\gamma}{2}}(Q-\beta_1)\Gamma_{\frac{\gamma}{2}}(Q-\beta_2)\Gamma_{\frac{\gamma}{2}}(\beta_3)}.
$$

The formula for H with two non-zero cosmological constants is given by:

$$
\begin{aligned}
\overline{H}^{(\beta_1,\beta_2,\beta_3)}_{(0,\mu_2,\mu_3)} &= \frac{(2\pi)^{\frac{2Q-\overline{\beta}}{\gamma}+1}(\frac{2}{\gamma})^{(\frac{\gamma}{2}-\frac{2}{\gamma})(Q-\frac{\overline{\beta}}{2})-1}}{\Gamma(1-\frac{\gamma^2}{4})^{\frac{2Q-\overline{\beta}}{\gamma}}\Gamma(\frac{\overline{\beta}-2Q}{\gamma})} \frac{\Gamma_{\frac{\gamma}{2}}(2Q-\frac{\overline{\beta}}{2})\Gamma_{\frac{\gamma}{2}}(\frac{\beta_1+\beta_3-\beta_2}{2})\Gamma_{\frac{\gamma}{2}}(Q-\frac{\beta_1+\beta_2-\beta_3}{2})\Gamma_{\frac{\gamma}{2}}(Q-\frac{\beta_2+\beta_3-\beta_1}{2})}{\Gamma_{\frac{\gamma}{2}}(Q)\Gamma_{\frac{\gamma}{2}}(Q-\beta_1)\Gamma_{\frac{\gamma}{2}}(Q-\beta_2)\Gamma_{\frac{\gamma}{2}}(Q-\beta_3)} \\
&\times e^{\mathbf{i}\pi\sigma_2(2Q-\beta_1-\beta_2-\beta_3)}e^{\frac{\mathbf{i}\pi}{2}Q(\beta_1+\beta_2+\beta_3-2Q)}\int_{\mathcal{C}} \frac{S_{\frac{\gamma}{2}}(\frac{\beta_2}{2}+\frac{\beta_3}{2}-\frac{\beta_1}{2}+r)S_{\frac{\gamma}{2}}(\frac{\overline{\beta}}{2}-Q+r)}{S_{\frac{\gamma}{2}}(Q+r)S_{\frac{\gamma}{2}}(\beta_3+r)}e^{-i2\pi(\sigma_2-\sigma_3)r}\frac{dr}{i}.
\end{aligned}
$$

This last integral is converging if and only if:

$$
\mathrm{Re}(\beta_2 - 2Q + 2(\sigma_2-\sigma_3)) < 0, \quad \mathrm{Re}(2Q-\beta_2+2(\sigma_2-\sigma_3)) > 0.
$$

2.4. **Conformal bootstrap for boundary LCFT.** The next step in solving Liouville CFT is known as the conformal bootstrap. This has been performed in the case of boundaryless surfaces in the breakthrough works [GKRV20, GKRV21]. In the case of surfaces with boundary, there is a result in the case of the annulus [Wu22] and a work in progress by the authors of [GKRV20, Wu22], in the case of general surfaces with boundary.

In order to derive the fusion transformation of the spherical conformal block, we will use the two different ways of writing the conformal bootstrap for the four-point boundary disk correlation function. These statements have not yet been proved in the literature but they are a work in preparation by the authors of [GKRV20, Wu22].

Theorem 2.6. Let $\alpha_i < Q$, $\alpha_1+\alpha_2 > Q$, $\alpha_3+\alpha_4>Q$ and $z\in(0,1)$. Then one has:

$$\left\langle \prod_{i=1}^4 B_{\alpha_i}^{\mu_i,\mu_{i+1}}(z_i)\right\rangle = z^{(\frac{Q^2}{4}-\Delta_{\alpha_1}-\Delta_{\alpha_2})}\int_{\mathbb{R}} H_{(\mu_1,\mu_2,\mu_3)}^{(\alpha_1,\alpha_2,Q+iP)} H_{(\mu_3,\mu_4,\mu_1)}^{(\alpha_3,\alpha_4,Q-iP)} z^{\frac{P^2}{4}}\mathcal{F}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(z,P)dP. \tag{2.8}$$

Here the integration over P is absolutely converging, i.e., it converges with absolute values added.

The previous statement corresponds to the fusion $z\to 0$. Let us now give the $z\to 1$ statement:

Theorem 2.7. Let $\alpha_i < Q$, $\alpha_1+\alpha_4 > Q$, $\alpha_2+\alpha_3>Q$ and $z\in(0,1)$. Then one has:

$$\left\langle \prod_{i=1}^4 B_{\alpha_i}^{\mu_i,\mu_{i+1}}(z_i)\right\rangle = (1-z)^{(\frac{Q^2}{4}-\Delta_{\alpha_3}-\Delta_{\alpha_2})}\int_{\mathbb{R}} H_{(\mu_1,\mu_2,\mu_4)}^{(\alpha_1,Q+iP',\alpha_4)} H_{(\mu_2,\mu_3,\mu_4)}^{(\alpha_2,\alpha_3,Q-iP')} (1-z)^{\frac{P'^2}{4}}\mathcal{F}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(1-z,P')dP'.$$

The integral over P is again absolutely converging.

2.5. **The 1-point torus case.** We give the analogue of the previous subsections required to prove our results for one-point torus conformal block. We will require LCFT and the bootstrap statements on an annulus. For $\tau\in\mathbf{i}\mathbb{R}$, we define the annulus

$$\mathcal{C}_\tau := \text{the rectangule bounded by } 0,1,\frac{\tau}{2},1+\frac{\tau}{2} \text{ with vertical edges identified.}$$

2.5.1. *Boundary Liouville theory on the annulus.* We will now define Liouville CFT on the annulus $\mathcal{C}_\tau$. We define the annulus boundaries

$$\partial_0\mathcal{C}_\tau := [0,1] \qquad \partial_1\mathcal{C}_\tau := [\tau/2, 1+\tau/2].$$

We adopt the notations of [ARS22] with $\tau_{\mathrm{ARS}} := \frac{\tau}{2\mathbf{i}}$. In particular, we define:

- LF_τ is the law of the Liouville field ϕ on $\mathcal{C}_\tau$, defined as in [ARS22, Definition 2.2] with τ_{ARS}.
- $\mathcal{L}^\gamma_\phi$ is the quantum length operator
- $L_0 := \mathcal{L}^\gamma_\phi(\partial_0\mathcal{C}_\tau)$ and $L_1 := \mathcal{L}^\gamma_\phi(\partial_1\mathcal{C}_\tau)$ are the boundary lengths.

In what follows, we fix the following cosmological constants:

- $\mu=0$, the bulk cosmological constant
- $\mu_0>0$, the boundary cosmological constant corresponding to $\partial_0\mathcal{C}_\tau$
- $\mu_1>0$, the boundary cosmological constant corresponding to $\partial_1\mathcal{C}_\tau$

2.5.2. *Boundary bootstrap on the annulus.* With the previous notations we can now state the bootstrap results on $\mathcal{C}_\tau$. The first boundary bootstrap, proved in [Wu22, Theorem 1.2], is the following statement, which we extrapolate from the case $\alpha=\gamma$ in [ARS22, Theorem 3.1].

Theorem 2.8. Let $\alpha\in(0,Q)$, $q\in(0,1)$. We have that:

$$\frac{1}{\eta(q)}\mathrm{LF}_\tau\left[L_1 e^{-\mu_0L_0-\mu_1L_1}V_{\frac{\alpha}{2}}(0)\right] = -\frac{1}{2\pi}\int_{\mathbb{R}} G_{\mu_0}(\alpha,Q+\mathbf{i}P)\partial_{\mu_1}U_{\mu_1}(Q-\mathbf{i}P)q^{\frac12P^2}q^{-\frac{1}{12}}\mathcal{F}^{\mathsf{torus}}_\alpha(q,P)dP.$$

In the work in preparation (cite), the following alternative boundary bootstrap which corresponds to a vertical cut of the annulus will be established.

Theorem 2.9. Let $\alpha\in(0,Q)$, $q\in(0,1)$. We have that

$$\frac{1}{\eta(q)}\mathrm{LF}_\tau\left[e^{-\mu_0L_0-\mu_1L_1}V_{\frac{\alpha}{2}}(0)\right] = \frac{C_1}{2\pi}\tau^{-\frac{\alpha}{2}(Q-\frac{\alpha}{2})}\int_{\mathbb{R}} H^{(\alpha,Q+\mathbf{i}P,Q-\mathbf{i}P)}_{(\mu_0,\mu_0,\mu_1)}\tilde{q}^{\frac12P^2}\tilde{q}^{-\frac{1}{12}}\mathcal{F}^{\mathsf{torus}}_\alpha(\tilde{q},P)dP,$$

for C_1 a global constant independent of all parameters.

3. GMC expression and fusion transformation of spherical conformal blocks

The main goal of this section is twofold. In Section 3.1, we provide an expression of the 4-point sphere conformal block using the Gaussian multiplicative chaos measure and relate it to the conformal block appearing in the bootstrap statements of Theorems 2.6 and 2.7. In Section 3.2, we prove Theorem 1.5 which shows the fusion transformation for the conformal block defined using GMC in the parameter range where it is well-defined.

3.1. **GMC expression for 4-point sphere conformal block.** We start by introducing the probabilistic definition of the 4-point sphere conformal block. The definition will be stated using Gaussian multiplicative chaos with respect to the GFF on $\mathbb{H}$ with insertions. Recall that h is the free boundary GFF on $\mathbb{H}$ (see Section 2.1 for its definition). Now we define the field $\hat{h}$ by adding some deterministic function with with h which is given below:

$$\hat{h}(x) = h(x) + \alpha_1 \log \frac{1}{|x|} + \alpha_2 \log \frac{1}{|x-z|} + \alpha_3 \log \frac{1}{|x-1|} + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - 2Q) \log |x|_+. \tag{3.1}$$

Now we use $\hat{h}$ to define the sphere conformal block. Recall the definition of $\mathcal{L}_{\hat{h}}(0,z)$ and $\mathcal{L}_{\hat{h}}(1,\infty)$ from Section 2.2.

Definition 3.1. (Probabilistic definition of the 4-point spherical block) Let $\gamma \in (0,2)$ and $z \in (0,1)$. Consider parameters $\alpha_1, \alpha_2, \alpha_3, \alpha_4 < Q$ and $P \in \mathbb{C}$ obeying the following bounds:

$$\frac{Q + \mathrm{Im}(P) - \alpha_1 - \alpha_2}{\gamma} < \frac{4}{\gamma^2} \wedge \frac{2}{\gamma}(Q-\alpha_1) \wedge \frac{2}{\gamma}(Q-\alpha_2), \quad \frac{Q - \mathrm{Im}(P) - \alpha_3 - \alpha_4}{\gamma} < \frac{4}{\gamma^2} \wedge \frac{2}{\gamma}(Q-\alpha_3) \wedge \frac{2}{\gamma}(Q-\alpha_4).$$

Our probabilistic definition of the 4-point sphere conformal block is given as

$$\mathcal{G}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(z,P) := \frac{z^{(\Delta_{\alpha_1}+\Delta_{\alpha_2}-\frac{P^2}{4}-\frac{Q^2}{4}-\frac{\alpha_1\alpha_2}{2})}(1-z)^{-\frac{\alpha_2\alpha_3}{2}}}{\mathcal{Z}} \mathbb{E}\left[\mathcal{L}_{\hat{h}}(0,z)^{\frac{Q-\mathbf{i}P-\alpha_1-\alpha_2}{\gamma}} \mathcal{L}_{\hat{h}}(1,\infty)^{\frac{Q+\mathbf{i}P-\alpha_3-\alpha_4}{\gamma}}\right],$$

where the normalization is given by

$$\mathcal{Z} = \frac{8\pi}{\gamma} \overline{H}^{(\alpha_1,\alpha_2,Q+\mathbf{i}P)}_{(0,1,0)} \overline{H}^{(\alpha_3,\alpha_4,Q-\mathbf{i}P)}_{(0,1,0)},$$

where the functions $\overline{H}$ have an explicit expression given in Lemma 2.5.

Remark. By using Lemma 2.5, the above normalization can be further evaluated as:

$$\mathcal{Z} = \frac{8\pi}{\gamma} \frac{(2\pi)^{\frac{2Q-\overline{\alpha}}{\gamma}+2} (\frac{2}{\gamma})^{(\frac{\gamma}{2}-\frac{2}{\gamma})(\frac{2Q-\overline{\alpha}}{2})-2}}{\Gamma(1-\frac{\gamma^2}{4})^{\frac{2Q-\overline{\alpha}}{\gamma}}} \frac{\Gamma_{\frac{\gamma}{2}}(\frac{\alpha_1+\alpha_2-Q+\mathbf{i}P}{2})\Gamma_{\frac{\gamma}{2}}(\frac{\alpha_1+Q+\mathbf{i}P-\alpha_2}{2})\Gamma_{\frac{\gamma}{2}}(\frac{\alpha_2+Q+\mathbf{i}P-\alpha_1}{2})\Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha_1+\alpha_2-Q-\mathbf{i}P}{2})}{\Gamma_{\frac{\gamma}{2}}(Q)\Gamma_{\frac{\gamma}{2}}(Q-\alpha_1)\Gamma_{\frac{\gamma}{2}}(Q-\alpha_2)\Gamma_{\frac{\gamma}{2}}(Q+\mathbf{i}P)}$$
$$\times \frac{\Gamma_{\frac{\gamma}{2}}(\frac{-Q+\alpha_3+\alpha_4-\mathbf{i}P}{2})\Gamma_{\frac{\gamma}{2}}(\frac{\alpha_3+Q-\mathbf{i}P-\alpha_4}{2})\Gamma_{\frac{\gamma}{2}}(\frac{\alpha_4+Q-\mathbf{i}P-\alpha_3}{2})\Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha_3+\alpha_4-Q+\mathbf{i}P}{2})}{\Gamma(\frac{\alpha_1+\alpha_2+\mathbf{i}P-Q}{\gamma})\Gamma(\frac{\alpha_3+\alpha_4-\mathbf{i}P-Q}{\gamma})\Gamma_{\frac{\gamma}{2}}(Q)\Gamma_{\frac{\gamma}{2}}(Q-\alpha_3)\Gamma_{\frac{\gamma}{2}}(Q-\alpha_4)\Gamma_{\frac{\gamma}{2}}(Q-\mathbf{i}P)}.$$

The following lemma will show the above definition of the probabilistic conformal block is well-posed in the given parameter range.

Lemma 3.2. In the parameter range of Definition 3.1, the following holds:

$$\mathbb{E}\left[\left|\mathcal{L}_{\hat{h}}(0,z)^{\frac{Q-\mathbf{i}P-\alpha_1-\alpha_2}{\gamma}} \mathcal{L}_{\hat{h}}(1,\infty)^{\frac{Q+\mathbf{i}P-\alpha_3-\alpha_4}{\gamma}}\right|\right] < +\infty.$$

Proof. Fix $z \in (0,1)$ and $z_0 \in (z,1)$. Consider the vertical line in $\mathbb{H}$ given by $L = \{z_0 + \mathbf{i}y, y > 0\}$. We can split the upper-half plane into $\mathbb{H} = \mathbb{H}_1 \cup \mathbb{H}_2 \cup L$, where $\mathbb{H}_1$ (respectively $\mathbb{H}_2$) is the subdomain of $\mathbb{H}$ to the left (respectively to the right) of the line L. We will then apply the Markov property of the GFF h to this decomposition, namely we write $h = h_1 + h_2 + h_L$, where h_1, h_2 are GFFs respectively on $\mathbb{H}_1, \mathbb{H}_2$ with zero boundary condition on L and h_L is the harmonic extension on $\mathbb{H}$ of the restriction of h to L. Note that h_1, h_2, h_L are independent and that h_L is smooth outside of L. We also denote by $\hat{h}_1, \hat{h}_2$ the law of h_1, h_2 weighted by the four insertions, namely defined using (3.1) but with $h_i, \hat{h}_i$ instead of $h, \hat{h}$. We now apply this decomposition to our GMC expression of the conformal block:

$$\mathbb{E}\left[\left|\mathcal{L}_{\hat{h}}(0,z)^{\frac{Q-\mathbf{i}P-\alpha_1-\alpha_2}{\gamma}} \mathcal{L}_{\hat{h}}(1,\infty)^{\frac{Q+\mathbf{i}P-\alpha_3-\alpha_4}{\gamma}}\right|\right] = \mathbb{E}\left[\mathcal{L}_{\hat{h}_1+h_L}(0,z)^{\frac{Q+\mathrm{Im}(P)-\alpha_1-\alpha_2}{\gamma}} \mathcal{L}_{\hat{h}_2+h_L}(1,\infty)^{\frac{Q-\mathrm{Im}(P)-\alpha_3-\alpha_4}{\gamma}}\right]$$
$$\leq \mathbb{E}\left[e^{|\frac{Q+\mathrm{Im}(P)-\alpha_1-\beta_2}{2}|\sup_{x\in(0,z)}|h_L(x)|} e^{|\frac{Q-\mathrm{Im}(P)-\alpha_3-\alpha_4}{2}|\sup_{x\in(1,\infty)}|h_L(x)|} \mathcal{L}_{\hat{h}_1}(0,z)^{\frac{Q+\mathrm{Im}(P)-\alpha_1-\alpha_2}{\gamma}} \mathcal{L}_{\hat{h}_2}(1,\infty)^{\frac{Q-\mathrm{Im}(P)-\alpha_3-\alpha_4}{\gamma}}\right]$$
$$\leq \mathbb{E}\left[e^{|\frac{Q+\mathrm{Im}(P)-\alpha_1-\alpha_2}{2}|\sup_{x\in(0,z)}|h_L(x)|} e^{|\frac{Q-\mathrm{Im}(P)-\alpha_3-\alpha_4}{2}|\sup_{x\in(1,\infty)}|h_L(x)|}\right]$$
$$\times \mathbb{E}\left[\mathcal{L}_{\hat{h}_1}(0,z)^{\frac{Q+\mathrm{Im}(P)-\alpha_1-\alpha_2}{\gamma}}\right] \mathbb{E}\left[\mathcal{L}_{\hat{h}_2}(1,\infty)^{\frac{Q-\mathrm{Im}(P)-\alpha_3-\alpha_4}{\gamma}}\right].$$

Above to obtain the second line we have bounded the field h_L inside each GMC by the maximum of its absolute value. Then in the last inequality we have used the independence of the three fields h_1, h_2, h_L. Now each of the three expectations appearing in this last line are finite. For the first it follows from the fact that h_L is a continuous Gaussian process away from the line L and therefore its maximum on any compact set admits exponential moments (here $\mathbb{R}$ is compact thanks to the background metric). For the last two terms the choice of parameters given in Definition 3.1 implies these two GMC moments are finite. This completes the proof. □

We now give a lemma computing the $z \to 0$ limit of this conformal block.

Remark. We can directly check on this expression that as $z \to 0$, the conformal block is equal to 1. Recall the expression of $\hat{h}$ given by (3.1). We now write out the two GMC integrals:

$$\mathcal{L}_{\hat{h}}(0,z) = \int_0^z |x|^{-\frac{\alpha_1\gamma}{2}}|x-z|^{-\frac{\alpha_2\gamma}{2}}|x-1|^{-\frac{\alpha_3\gamma}{2}}e^{\frac{\gamma}{2}h(x)}dx,$$
$$\mathcal{L}_{\hat{h}}(1,\infty) = \int_1^\infty |x|^{\frac{\gamma}{2}(\alpha_2+\alpha_3+\alpha_4-2Q)}|x-z|^{-\frac{\alpha_2\gamma}{2}}|x-1|^{-\frac{\alpha_3\gamma}{2}}e^{\frac{\gamma}{2}h(x)}dx.$$

We take the limit $z \to 0$. In the first GMC we will use the scaling of the field, for $u \in (0,1)$, $z \in (0,1)$:

$$e^{\frac{\gamma}{2}h(zu)}du = |z|^{\frac{\gamma^2}{4}}e^{\frac{\gamma}{2}\mathcal{N}(2\log\frac{1}{|z|})}e^{\frac{\gamma}{2}h(u)}du.$$

This gives:

$$\mathcal{L}_{\hat{h}}(0,z) = z\int_0^1 |zu|^{-\frac{\alpha_1\gamma}{2}}|zu-z|^{-\frac{\alpha_2\gamma}{2}}|zu-1|^{-\frac{\alpha_3\gamma}{2}}e^{\frac{\gamma}{2}h(zu)}du$$
$$\underset{z\to 0}{\sim} z^{\frac{\gamma}{2}(Q-\alpha_1-\alpha_2)}e^{\frac{\gamma}{2}\mathcal{N}(2\log\frac{1}{|z|})}\int_0^1 |u|^{-\frac{\alpha_1\gamma}{2}}|u-1|^{-\frac{\alpha_2\gamma}{2}}e^{\frac{\gamma}{2}h(u)}du.$$

For the second GMC, using the change of variable $x = \frac{1}{u}$ and the equality in law $h(1/u) = h(u)$ we write:

$$\mathcal{L}_{\hat{h}}(1,\infty) \underset{z\to 0}{\sim} \int_1^\infty |x|^{\frac{\gamma}{2}(\alpha_3+\alpha_4-2Q)}|x-1|^{-\frac{\alpha_3\gamma}{2}}e^{\frac{\gamma}{2}h(x)}dx = \int_0^1 |x|^{-\frac{\alpha_4\gamma}{2}}|u-1|^{-\frac{\alpha_3\gamma}{2}}e^{\frac{\gamma}{2}h(u)}du.$$

Using the fact that in limit the two GMC become independent we get that:

$$\mathbb{E}\left[\mathcal{L}_{\hat{h}}(0,z)^{\frac{Q-\mathbf{i}P-\alpha_1-\alpha_2}{\gamma}}\mathcal{L}_{\hat{h}}(1,\infty)^{\frac{Q+\mathbf{i}P-\alpha_3-\alpha_4}{\gamma}}\right] \underset{z\to 0}{\sim} z^{(Q-\alpha_1-\alpha_2)\frac{Q-\mathbf{i}P-\alpha_1-\alpha_2}{2}}\mathbb{E}\left[e^{\frac{Q-\mathbf{i}P-\alpha_1-\alpha_2}{2}\mathcal{N}(2\log\frac{1}{|z|})}\right]$$
$$\times\mathbb{E}\left[\left(\int_0^1 |u|^{-\frac{\alpha_1\gamma}{2}}|u-1|^{-\frac{\alpha_2\gamma}{2}}e^{\frac{\gamma}{2}h(u)}du\right)^{\frac{Q-\mathbf{i}P-\alpha_1-\alpha_2}{\gamma}}\right]\mathbb{E}\left[\left(\int_0^1 |x|^{-\frac{\alpha_4\gamma}{2}}|u-1|^{-\frac{\alpha_3\gamma}{2}}e^{\frac{\gamma}{2}h(u)}du\right)^{\frac{Q+\mathbf{i}P-\alpha_3-\alpha_4}{\gamma}}\right]$$
$$= z^{\frac{Q^2}{4}+\frac{P^2}{4}+\frac{\alpha_1\alpha_2}{2}-\Delta_{\alpha_1}-\Delta_{alpha_2}}\overline{H}^{(\alpha_1,\alpha_2,Q+\mathbf{i}P)}_{(0,1,0)}\overline{H}^{(\alpha_4,\alpha_3,Q-\mathbf{i}P)}_{(0,1,0)}.$$

We can now evaluate the product of GMC's by the result of [RZ21] recalled in Lemma 2.5.

We now state the main result of this section which gives the almost everywhere equality of the two definitions of conformal block, namely the series expression given in Section 1.1 and the probabilistic expression introduced in Definition 3.1.

Theorem 3.3. Let α_i be such that $\alpha_i < Q$, $\alpha_1+\alpha_2 > Q$, $\alpha_3+\alpha_4 > Q$. Fix $z \in (0,1)$. Then one has:

$$\mathcal{F}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(z,P) = \mathcal{G}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(z,P) \quad \text{almost everywhere in } P \in \mathbb{R}.$$

We will use the bootstrap statement of Theorem 2.6 to prove Theorem 3.3. Since the 4-point sphere conformal block is free of the parameters $\{\mu_i\}_{1\le i\le 4}$ of Theorem 2.6, we are free to choose the μ_i as we wish. Throughout the whole proof below we will make the following choice:

$$\mu_1 = \mu_3 = 0, \quad \mu_4 = 1.$$

The main idea for proving Theorem 3.3 is to apply the operator $\int_0^\infty d\mu_2\mu_2^{-a-1}$ to both sides of the bootstrap statement of Theorem 2.6, where $a = \frac{Q-\mathbf{i}P-\alpha_1-\alpha_2}{\gamma}$. When applied to the correlation function in the left hand side of Theorem 2.6 this operator will recover $\mathcal{G}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(z,P)$ and when applied to the bootstrap integral in the right hand side it will recover $\mathcal{F}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(z,P)$. The following lemma performs the first of these two steps.

Lemma 3.4. Let $a = \frac{Q-\mathbf{i}P-\alpha_1-\alpha_2}{\gamma}$. In the parameter range

$$\alpha_i < Q, \quad \alpha_1+\alpha_2 > Q + \mathrm{Im}(P), \quad \alpha_3+\alpha_4 > Q - \mathrm{Im}(P), \tag{3.2}$$

the following identity holds:

$$\begin{aligned}
&\int_0^\infty d\mu_2 \mu_2^{-a-1} \int e^{-\mu_2 \mathcal{L}_\phi(0,z) - \mathcal{L}_\phi(1,\infty)} \, \mathrm{LF}_{\mathbb{H}}^{(\alpha_1,0),(\alpha_2,z),(\alpha_3,1),(\alpha_4,\infty)}(\mathrm{d}\phi) \\
&= |z|^{-\frac{\alpha_1\alpha_2}{2}} |1-z|^{-\frac{\alpha_2\alpha_3}{2}} \Gamma(\frac{\alpha_1+\alpha_2+\mathbf{i}P-Q}{\gamma}) \Gamma(\frac{\alpha_3+\alpha_4-\mathbf{i}P-Q}{\gamma}) \mathbb{E}\left[\mathcal{L}_{\hat h}(0,z)^{\frac{Q-\mathbf{i}P-\alpha_1-\alpha_2}{\gamma}} \mathcal{L}_{\hat h}(1,\infty)^{\frac{Q+\mathbf{i}P-\alpha_3-\alpha_4}{\gamma}}\right] \\
&= z^{(-\Delta_{\alpha_1}-\Delta_{\alpha_2}+\frac{P^2}{4}+\frac{Q^2}{4})} \Gamma(\frac{\alpha_1+\alpha_2+\mathbf{i}P-Q}{\gamma}) \Gamma(\frac{\alpha_3+\alpha_4-\mathbf{i}P-Q}{\gamma}) \mathcal{Z}\mathcal{G}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(z,P).
\end{aligned} \tag{3.3}$$

Proof. Recall from (3.1) that $\hat h$ is the GFF weighted by the four insertions. Let us write out explicitly the integration over the zero mode in the definition of the correlation function:

$$\begin{aligned}
&\int e^{-\mu_2 \mathcal{L}_\phi(0,z) - \mathcal{L}_\phi(1,+\infty)} \, \mathrm{LF}_{\mathbb{H}}^{(\alpha_1,0),(\alpha_2,z),(\alpha_3,1),(\alpha_4,\infty)}(\mathrm{d}\phi) \\
&= |z|^{-\frac{\alpha_1\alpha_2}{2}} |1-z|^{-\frac{\alpha_2\alpha_3}{2}} \int_{\mathbb{R}} dc e^{\frac{1}{2}(\alpha_1+\alpha_2+\alpha_3+\alpha_4-2Q)c} \mathbb{E}\left[e^{-e^{\frac{\gamma c}{2}}(\mu_2\mathcal{L}_{\hat h}(0,z)+\mathcal{L}_{\hat h}(1,\infty))}\right] \\
&= |z|^{-\frac{\alpha_1\alpha_2}{2}} |1-z|^{-\frac{\alpha_2\alpha_3}{2}} \Gamma(\frac{\alpha_1+\alpha_2+\alpha_3+\alpha_4-2Q}{\gamma}) \mathbb{E}\left[\left(\mu_2\mathcal{L}_{\hat h}(0,z)+\mathcal{L}_{\hat h}(1,\infty)\right)^{\frac{2Q-\alpha_1-\alpha_2-\alpha_3-\alpha_4}{\gamma}}\right].
\end{aligned} \tag{3.4}$$

Notice that (3.2) implies $\sum_{i=1}^4 \alpha_i > 2Q$, which implies the above moment of GMC is a negative moment and is thus finite for any $\mu_2 \in (0,\infty)$. Now we claim and prove that

$$\begin{aligned}
&\int_0^\infty d\mu_2 \mu_2^{\frac{\alpha_1+\alpha_2+\mathbf{i}P-Q}{\gamma}-1} \mathbb{E}\left[\left(\mu_2\mathcal{L}_{\hat h}(0,z)+\mathcal{L}_{\hat h}(1,\infty)\right)^{\frac{2Q-\alpha_1-\alpha_2-\alpha_3-\alpha_4}{\gamma}}\right] \\
&\qquad = \frac{\Gamma(\frac{\alpha_1+\alpha_2+\mathbf{i}P-Q}{\gamma})\Gamma(\frac{\alpha_3+\alpha_4-\mathbf{i}P-Q}{\gamma})}{\Gamma(\frac{\alpha_1+\alpha_2+\alpha_3+\alpha_4-2Q}{\gamma})} \mathbb{E}\left[\mathcal{L}_{\hat h}(0,z)^{\frac{Q-\alpha_1-\alpha_2-\mathbf{i}P}{\gamma}} \mathcal{L}_{\hat h}(1,\infty)^{\frac{Q+\mathbf{i}P-\alpha_3-\alpha_4}{\gamma}}\right]
\end{aligned} \tag{3.5}$$

Before proceeding to the proof of the above identity, observe that substituting this identity into the right hand side of (3.4) implies (3.2). Therefore it suffices to show the above identity.

We first show that the integral in the right hand side of (3.5) is well defined. To this end, we show that integral of the absolute of the corresponding integrand is finite. Taking the absolute value leads to transform the left hand side of (3.5) into the following expression:

$$\begin{aligned}
&\int_0^\infty d\mu_2 \mu_2^{\frac{\alpha_1+\alpha_2-\mathrm{Im}(P)-Q}{\gamma}-1} \mathbb{E}\left[\left(\mu_2\mathcal{L}_{\hat h}(0,z)+\mathcal{L}_{\hat h}(1,\infty)\right)^{\frac{2Q-\alpha_1-\alpha_2-\alpha_3-\alpha_4}{\gamma}}\right] \\
&= \int_0^\infty du \, u^{\frac{\alpha_1+\alpha_2-\mathrm{Im}(P)-Q}{\gamma}-1} \mathbb{E}\left[\mathcal{L}_{\hat h}(0,z)^{\frac{\mathrm{Im}(P)+Q-\alpha_1-\alpha_2}{\gamma}} \left(u+\mathcal{L}_{\hat h}(1,\infty)\right)^{\frac{2Q-\alpha_1-\alpha_2-\alpha_3-\alpha_4}{\gamma}}\right]
\end{aligned}$$

where the equality follows by making a change of variable $\mu_2\mathcal{L}_{\hat h}(0,z) \mapsto u$. We will now show that the last line of the above display is finite. Applying the Hölder inequality to the last line of the above display yields

$$\begin{aligned}
&\int_0^\infty du \, u^{\frac{\alpha_1+\alpha_2-\mathrm{Im}(P)-Q}{\gamma}-1} \mathbb{E}\left[\mathcal{L}_{\hat h}(0,z)^{\frac{\mathrm{Im}(P)+Q-\alpha_1-\alpha_2}{\gamma}} \left(u+\mathcal{L}_{\hat h}(1,\infty)\right)^{\frac{2Q-\alpha_1-\alpha_2-\alpha_3-\alpha_4}{\gamma}}\right] \\
&\le \mathbb{E}\left[\mathcal{L}_{\hat h}(0,z)^{\frac{(\mathrm{Im}(P)+Q-\alpha_1-\alpha_2)p_1}{\gamma}}\right]^{\frac{1}{p_1}} \int_0^\infty du \, u^{\frac{\alpha_1+\alpha_2-\mathrm{Im}(P)-Q}{\gamma}-1} \mathbb{E}\left[\left(u+\mathcal{L}_{\hat h}(1,\infty)\right)^{\frac{(2Q-\alpha_1-\alpha_2-\alpha_3-\alpha_4)p_2}{\gamma}}\right]^{\frac{1}{p_2}}.
\end{aligned}$$

Here $p_1, p_2 > 1$ are such that $\frac{1}{p_1}+\frac{1}{p_2} = 1$ and p_2 arbitrarily close to 1. Recall our condition $\alpha_1+\alpha_2 > Q+\mathrm{Im}(P)$. This implies the first term in the last line of the above display is a negative moment of GMC and hence, the first term is finite for any $p_1 > 1$. For the second term, we first write the following by making a change of variable $u/\mathcal{L}_{\hat{h}}(1,\infty) \mapsto v$:

$$\int_0^\infty du\, u^{\frac{\alpha_1+\alpha_2-\mathrm{Im}(P)-Q}{\gamma}-1}\mathbb{E}\left[\left(u+\mathcal{L}_{\hat{h}}(1,\infty)\right)^{\frac{(2Q-\alpha_1-\alpha_2-\alpha_3-\alpha_4)p_2}{\gamma}}\right]^{\frac{1}{p_2}}$$
$$=\mathbb{E}\left[\left(\mathcal{L}_{\hat{h}}(1,\infty)\right)^{\frac{(Q-\mathrm{Im}(P)-\alpha_3-\alpha_4)p_2}{\gamma}}\right]^{\frac{1}{p_2}}\int_0^\infty du\, v^{\frac{\alpha_1+\alpha_2-\mathrm{Im}(P)-Q}{\gamma}-1}(v+1)^{\frac{(2Q-\alpha_1-\alpha_2-\alpha_3-\alpha_4)}{\gamma}}.$$

In the above expression, the moment of GMC is negative and is thus finite, and the integral is converging. Therefore the integral of (3.5) is converging. Making the change of variable $\mu_2\mathcal{L}_{\hat{h}}(0,z)/\mathcal{L}_{\hat{h}}(1,\infty)\mapsto\mu$ in our integral of interest yields

$$\int_0^\infty d\mu_2\mu_2^{\frac{\alpha_1+\alpha_2+\mathbf{i}P-Q}{\gamma}-1}\mathbb{E}\left[\left(\mu_2\mathcal{L}_{\hat{h}}(0,z)+\mathcal{L}_{\hat{h}}(1,\infty)\right)^{\frac{2Q-\alpha_1-\alpha_2-\alpha_3-\alpha_4}{\gamma}}\right]$$
$$=\int_0^\infty d\mu\mu^{\frac{\alpha_1+\alpha_2+\mathbf{i}P-Q}{\gamma}-1}(\mu+1)^{\frac{2Q-\alpha_1-\alpha_2-\alpha_3-\alpha_4}{\gamma}}\mathbb{E}\left[\mathcal{L}_{\hat{h}}(0,z)^{\frac{Q-\alpha_1-\alpha_2-\mathbf{i}P}{\gamma}}\mathcal{L}_{\hat{h}}(1,\infty)^{\frac{Q+\mathbf{i}P-\alpha_3-\alpha_4}{\gamma}}\right]$$
$$=\frac{\Gamma(\frac{\alpha_1+\alpha_2+\mathbf{i}P-Q}{\gamma})\Gamma(\frac{\alpha_3+\alpha_4-\mathbf{i}P-Q}{\gamma})}{\Gamma(\frac{\alpha_1+\alpha_2+\alpha_3+\alpha_4-2Q}{\gamma})}\mathbb{E}\left[\mathcal{L}_{\hat{h}}(0,z)^{\frac{Q-\alpha_1-\alpha_2-\mathbf{i}P}{\gamma}}\mathcal{L}_{\hat{h}}(1,\infty)^{\frac{Q+\mathbf{i}P-\alpha_3-\alpha_4}{\gamma}}\right].$$

This shows (3.5) and hence, proves the claimed result. □

We now finish the proof of Theorem 3.3.

Proof of Theorem 3.3. Recall the definition of $\mathcal{Z}$ from Definition 3.1. It was written in terms of $\overline{H}^{(\alpha_1,\alpha_2,Q+\mathbf{i}P)}_{(0,1,0)}$ and $\overline{H}^{(\alpha_3,\alpha_4,Q-\mathbf{i}P)}_{(0,1,0)}$. Using the relation between $\overline{H}$ and H as written in Lemma 2.5, we observe

$$\mathcal{Z}=2\pi\gamma\Gamma(\frac{\alpha_1+\alpha_2+\mathbf{i}P-Q}{\gamma})^{-1}\Gamma(\frac{\alpha_3+\alpha_4-\mathbf{i}P-Q}{\gamma})^{-1}H^{(\alpha_1,\alpha_2,Q+\mathbf{i}P)}_{(0,1,0)}H^{(\alpha_3,\alpha_4,Q-\mathbf{i}P)}_{(0,1,0)}.$$

Hence, to complete the proof of our desired result, it suffices to show

$$H^{(\alpha_1,\alpha_2,Q+iP)}_{(0,1,0)}H^{(\alpha_3,\alpha_4,Q-iP)}_{(0,1,0)}\mathcal{F}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(z,P)dP \tag{3.6}$$
$$=\frac{1}{2\pi\gamma}\Gamma(\frac{\alpha_1+\alpha_2+\mathbf{i}P-Q}{\gamma})\Gamma(\frac{\alpha_3+\alpha_4-\mathbf{i}P-Q}{\gamma})\mathcal{Z}\mathcal{G}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(z,P)\quad\text{a.e.} \tag{3.7}$$

We prove this by using the identity in (2.8) of Theorem 2.6 and the inverse Fourier transform. Note that the right hand side of (2.8) depends on μ_2 only through one of the H functions, which by a simple scaling argument (recalled in Lemma 2.5) can be written as:

$$z^{(\frac{Q^2}{4}-\Delta_{\alpha_1}-\Delta_{\alpha_2})}\int_{\mathbb{R}}H^{(\alpha_1,\alpha_2,Q+iP)}_{(0,\mu_2,0)}H^{(\alpha_3,\alpha_4,Q-iP)}_{(0,1,0)}z^{\frac{P^2}{4}}\mathcal{F}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(z,P)dP$$
$$=z^{(\frac{Q^2}{4}-\Delta_{\alpha_1}-\Delta_{\alpha_2})}\int_{\mathbb{R}}\mu_2^{\frac{Q-\alpha_1-\alpha_2-\mathbf{i}P'}{\gamma}}H^{(\alpha_1,\alpha_2,Q+iP)}_{(0,1,0)}H^{(\alpha_3,\alpha_4,Q-iP)}_{(0,1,0)}z^{\frac{P^2}{4}}\mathcal{F}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(z,P)dP.$$

On the other hand Lemma 3.4 tells us that (recall here $a=\frac{Q-\mathbf{i}P-\alpha_1-\alpha_2}{\gamma}$):

$$\int_0^\infty d\mu_2\mu_2^{-a-1}\int e^{-\mu_2\mathcal{L}_\phi(0,z)-\mathcal{L}_\phi(1,\infty)}\,\mathrm{LF}_{\mathbb{H}}^{(\alpha_1,0),(\alpha_2,z),(\alpha_3,1),(\alpha_4,\infty)}(\mathrm{d}\phi)$$
$$=z^{(-\Delta_{\alpha_1}-\Delta_{\alpha_2}+\frac{P^2}{4}+\frac{Q^2}{4})}\Gamma(\frac{\alpha_1+\alpha_2+\mathbf{i}P-Q}{\gamma})\Gamma(\frac{\alpha_3+\alpha_4-\mathbf{i}P-Q}{\gamma})\mathcal{Z}\mathcal{G}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(z,P). \tag{3.8}$$

Recall the following Fourier inversion identity which holds for any continuous $f\in L^1(\mathbb{R})$ with Fourier transform also in $L^1(\mathbb{R})$, after substituting $\mu_2=e^\omega$, $\tilde{\mu}_2=e^{\omega'}$:

$$\int_{\mathbb{R}}dP\tilde{\mu}_2^{\frac{Q-\mathbf{i}P-\alpha_1-\alpha_2}{\gamma}}\int_0^\infty d\mu_2\mu_2^{\frac{-Q+\alpha_1+\alpha_2+\mathbf{i}P}{\gamma}-1}f(\mu_2)$$
$$=\int_{\mathbb{R}}dP\int_{\mathbb{R}}d\omega e^{(\omega'-\omega)(\frac{Q-\mathbf{i}P-\alpha_1-\alpha_2}{\gamma})}f(e^\omega)=2\pi\gamma f(e^{\omega'})=2\pi\gamma f(\tilde{\mu}_2).$$

Multiplying both sides of (3.8) by $(2\pi\gamma)^{-1}\tilde{\mu}_2^{(Q-\alpha_1-\alpha_2-\mathbf{i}P)/\gamma}$, integrating w.r.t. P over $\mathbb{R}$ and applying the above Fourier transform identity yields

$$\int e^{-\tilde{\mu}_2\mathcal{L}_\phi(0,z)-\mathcal{L}_\phi(1,\infty)}\ \mathrm{LF}_{\mathbb{H}}^{(\alpha_1,0),(\alpha_2,z),(\alpha_3,1),(\alpha_4,\infty)}(\mathrm{d}\phi)$$
$$= \frac{1}{2\pi\gamma}\int_{\mathbb{R}} dP\tilde{\mu}_2^{\frac{Q-\alpha_1-\alpha_2-\mathbf{i}P}{\gamma}} z^{(-\Delta_{\alpha_1}-\Delta_{\alpha_2}+\frac{P^2}{4}+\frac{Q^2}{4})}\Gamma(\frac{\alpha_1+\alpha_2+\mathbf{i}P-Q}{\gamma})\Gamma(\frac{\alpha_3+\alpha_4-\mathbf{i}P-Q}{\gamma})\mathcal{Z}\mathcal{G}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(z,P).$$

Notice that left hand side of the above display is boundary four point function as defined in Definition 2.3 with parameters $0, \tilde{mu}_2, 0, 1$. By applying the boundary bootstrap relation of Theorem 2.6 and the scaling relation of Lemma 2.5, we may write the above identity as:

$$z^{(\frac{Q^2}{4}-\Delta_{\alpha_1}-\Delta_{\alpha_2})}\int_{\mathbb{R}}\tilde{\mu}_2^{\frac{Q-\alpha_1-\alpha_2-\mathbf{i}P}{\gamma}} H^{(\alpha_1,\alpha_2,Q+iP)}_{(0,1,0)} H^{(\alpha_3,\alpha_4,Q-iP)}_{(0,1,0)} z^{\frac{P^2}{4}}\mathcal{F}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(z,P)dP$$
$$= \frac{1}{2\pi\gamma}\int_{\mathbb{R}} dP\tilde{\mu}_2^{\frac{Q-\alpha_1-\alpha_2-\mathbf{i}P}{\gamma}} z^{(-\Delta_{\alpha_1}-\Delta_{\alpha_2}+\frac{P^2}{4}+\frac{Q^2}{4})}\Gamma(\frac{\alpha_1+\alpha_2+\mathbf{i}P-Q}{\gamma})\Gamma(\frac{\alpha_3+\alpha_4-\mathbf{i}P-Q}{\gamma})\mathcal{Z}\mathcal{G}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(z,P).$$

We know that both integrals in the identity above are absolutely convergent. By applying the result that two L^1 function which have the same Fourier transform are almost everywhere equal we obtain (3.6). This completes the proof of Theorem 3.3. □

3.2. **Fusion transformation of the spherical conformal block.** In this section we will prove the fusion transformation for the spherical conformal block in a smaller parameter range than the one of our main result Theorem 1.2. The result is stated here using the GMC expression.

Theorem 3.5. Let α_i satisfy $\alpha_i < Q$, $\alpha_1+\alpha_2 > Q$, $\alpha_3+\alpha_4 > Q$, $\alpha_1+\alpha_4 > Q$, $\alpha_2+\alpha_3 > Q$. Fix $z \in (0,1)$ and $P \in \mathbb{R}$. Then the following holds

$$z^{\frac{1}{2}P^2}\mathcal{G}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(z,P) = C(z)\int_{\mathbb{R}}\mathcal{M}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(P,P')(1-z)^{\frac{1}{2}(P')^2}\mathcal{G}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(1-z,P')dP', \tag{3.9}$$

where $C(z) = z^{(\Delta_{\alpha_1}+\Delta_{\alpha_2}-\frac{Q^2}{2})}(1-z)^{(\frac{Q^2}{2}-\Delta_{\alpha_3}-\Delta_{\alpha_2})}$ and where the fusion kernel is given by

(3.10)

$$\mathcal{M}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(P,P')$$
$$= \frac{1}{2\pi}\mathcal{I}_1 \times \frac{\Gamma_{\frac{\gamma}{2}}(Q-\mathbf{i}P)\Gamma_{\frac{\gamma}{2}}(Q+\mathbf{i}P)\Gamma_{\frac{\gamma}{2}}(\frac{\alpha_1+\alpha_4-Q-\mathbf{i}P'}{2})\Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha_1+Q+\mathbf{i}P'-\alpha_4}{2})\Gamma_{\frac{\gamma}{2}}(Q-\frac{Q+\mathbf{i}P'+\alpha_4-\alpha_1}{2})}{\Gamma_{\frac{\gamma}{2}}(\frac{\alpha_1+\alpha_2-Q+\mathbf{i}P}{2})\Gamma_{\frac{\gamma}{2}}(\frac{\alpha_1+Q+\mathbf{i}P-\alpha_2}{2})\Gamma_{\frac{\gamma}{2}}(\frac{\alpha_2+Q+\mathbf{i}P-\alpha_1}{2})\Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha_1+\alpha_2-Q-\mathbf{i}P}{2})\Gamma_{\frac{\gamma}{2}}(-\mathbf{i}P')}$$
$$\times \frac{\Gamma_{\frac{\gamma}{2}}(\frac{3}{2}Q-\frac{\alpha_1+\alpha_4+\mathbf{i}P'}{2})\Gamma_{\frac{\gamma}{2}}(\frac{3}{2}Q-\frac{\alpha_2+\alpha_3-\mathbf{i}P'}{2})\Gamma_{\frac{\gamma}{2}}(\frac{\alpha_2+Q-\mathbf{i}P'-\alpha_3}{2})\Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha_2+\alpha_3-Q+\mathbf{i}P'}{2})\Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha_3+Q-\mathbf{i}P'-\alpha_2}{2})}{\Gamma_{\frac{\gamma}{2}}(\mathbf{i}P')\Gamma_{\frac{\gamma}{2}}(\frac{-Q+\alpha_3+\alpha_4-\mathbf{i}P}{2})\Gamma_{\frac{\gamma}{2}}(\frac{\alpha_3+Q-\mathbf{i}P-\alpha_4}{2})\Gamma_{\frac{\gamma}{2}}(\frac{\alpha_4+Q-\mathbf{i}P-\alpha_3}{2})\Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha_3+\alpha_4-Q+\mathbf{i}P}{2})},$$

where $\mathcal{I}_1$ has the integral expression:

$$\mathcal{I}_1 := \int_{\mathcal{C}} \frac{S_{\frac{\gamma}{2}}(\frac{Q+\mathbf{i}P'}{2}+\frac{\alpha_4}{2}-\frac{\alpha_1}{2}+r)S_{\frac{\gamma}{2}}(\frac{\alpha_1+\alpha_4+\mathbf{i}P'}{2}-\frac{Q}{2}+r)S_{\frac{\gamma}{2}}(\frac{Q}{2}-\frac{\alpha_3}{2}+r-\frac{\mathbf{i}P}{2}+\frac{\alpha_4}{2})S_{\frac{\gamma}{2}}(\frac{\alpha_3}{2}-\frac{Q}{2}+r-\frac{\mathbf{i}P}{2}+\frac{\alpha_4}{2})}{S_{\frac{\gamma}{2}}(Q+r)S_{\frac{\gamma}{2}}(\alpha_4+r)S_{\frac{\gamma}{2}}(Q+r-\frac{\mathbf{i}P}{2}+\frac{\mathbf{i}P'}{2}+\frac{\alpha_4}{2}-\frac{\alpha_2}{2})S_{\frac{\gamma}{2}}(\frac{\alpha_2}{2}+r-\frac{\mathbf{i}P}{2}+\frac{\mathbf{i}P'}{2}+\frac{\alpha_4}{2})}\frac{dr}{i}.$$

The above contour integral is well-defined in the chosen parameter range, see appendix A.2.

Will we prove this result by applying the same operator $\int_0^\infty d\mu_2\mu_2^{-a-1}$ to the right hand side of the second bootstrap statement given by Theorem 2.7. We first state the following lemma which will make the fusion kernel appear by applying the operator $\int_0^\infty d\mu_2\mu_2^{-a-1}$ to the H function in the right hand side of Theorem 2.7.

Lemma 3.6. Let again $a = \frac{Q-\mathbf{i}P-\alpha_1-\alpha_2}{\gamma}$. Let also α_i satisfy $\alpha_i < Q$, $\alpha_1+\alpha_2 > Q$, $\alpha_3+\alpha_4 > Q$, $\alpha_1+\alpha_4 > Q$, $\alpha_2+\alpha_3 > Q$. The following identity holds:

$$\int_0^\infty d\mu_2\mu_2^{-a-1}H^{(\alpha_1,Q+\mathbf{i}P',\alpha_4)}_{(0,\mu_2,1)}H^{(\alpha_2,\alpha_3,Q-\mathbf{i}P')}_{(\mu_2,0,1)} = \mathcal{Z}\Gamma(\frac{\alpha_1+\alpha_2+\mathbf{i}P-Q}{\gamma})\Gamma(\frac{\alpha_3+\alpha_4-\mathbf{i}P-Q}{\gamma})\mathcal{M}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(P,P').$$

Proof. Recall the relation lets start by writing out explicitly the relation $\mu_2 = e^{\mathbf{i}\pi\gamma(\sigma_2 - \frac{Q}{2})}$. We start by writing out explicitly the formula for the first H function (keeping here only the integral):

$$\int_0^\infty d\mu_2 \mu_2^{-a-1} e^{\pi \mathbf{i}\sigma_2(Q-\mathbf{i}P'-\alpha_1-\alpha_4)} \left(\int_{\mathcal{C}} \frac{S_{\frac{\gamma}{2}}(\frac{Q+\mathbf{i}P'}{2}+\frac{\alpha_4}{2}-\frac{\alpha_1}{2}+r) S_{\frac{\gamma}{2}}(\frac{\alpha_1+\alpha_4+\mathbf{i}P'}{2}-\frac{Q}{2}+r)}{S_{\frac{\gamma}{2}}(Q+r) S_{\frac{\gamma}{2}}(\alpha_4+r)} e^{-i2\pi(\sigma_2-\frac{Q}{2})r} \frac{dr}{i} \right) H^{(\alpha_1, Q+\mathbf{i}P'\alpha_4)}_{(0,\mu_2,1)}.$$

Lets first justify that we can exchange the integration over μ_2 and the integration over r, we need to check that:

$$\int_{\mathbf{i}\mathbb{R}} d\sigma_2 \int_{\mathcal{C}} dr \left| e^{\mathbf{i}\pi\sigma_2(\alpha_2-\alpha_4)} e^{i\pi Q r} \frac{S_{\frac{\gamma}{2}}(\frac{Q+\mathbf{i}P'}{2}+\frac{\alpha_4}{2}-\frac{\alpha_1}{2}+r) S_{\frac{\gamma}{2}}(\frac{\alpha_1+\alpha_4+\mathbf{i}P'}{2}-\frac{Q}{2}+r)}{S_{\frac{\gamma}{2}}(Q+r) S_{\frac{\gamma}{2}}(\alpha_4+r)} H^{(\alpha_1, Q+\mathbf{i}P'\alpha_4)}_{(0,\mu_2,1)} \right| < +\infty.$$

Note that after taking absolute values, the integrations over r and σ_2 decouple since the term $e^{-2\mathbf{i}\pi\sigma_2 r}$ is just a phase. By Lemma A.2, the integration over r is finite provided that:

$$-\mathrm{Im}(P') - 2Q < 0, \quad \mathrm{Im}(P') > 0.$$

We assume that P' has a small positive imaginary part in order for this integral to converge. By continuity we can take $\mathrm{Im}(P') \to 0$ at the end of the proof. Next we check the integral over σ_2. We use the fact that $H^{(\alpha_1, Q+\mathbf{i}P'\alpha_4)}_{(0,\mu_2,1)}$ converges to a constant as $\mu_2 \to 0$ and is equivalent to constant times $\mu_2^{\frac{Q-\mathbf{i}P'-\alpha_1-\alpha_4}{\gamma}}$ as $\mu_2 \to +\infty$. Now that we can interchange the order of integration we need to compute:

$$\int_{\mathbf{i}\mathbb{R}} d\sigma_2 e^{\pi\mathbf{i}\sigma_2(\mathbf{i}P-\mathbf{i}P'+\alpha_2-\alpha_4)} e^{-2\mathbf{i}\pi\sigma_2 r} H^{(\alpha_1, Q+\mathbf{i}P'\alpha_4)}_{(0,\mu_2,1)}. \tag{3.11}$$

For this we now write out the explicit for the second H function:

$$\int_{\mathcal{C}} \frac{S_{\frac{\gamma}{2}}(\frac{Q-\mathbf{i}P'}{2}+\frac{\alpha_2}{2}-\frac{\alpha_3}{2}+r') S_{\frac{\gamma}{2}}(\frac{\alpha_3+\alpha_2-\mathbf{i}P'}{2}-\frac{Q}{2}+r')}{S_{\frac{\gamma}{2}}(Q+r') S_{\frac{\gamma}{2}}(\alpha_2+r')} e^{-i2\pi(\frac{Q}{2}-\sigma_2)r'} \frac{dr'}{i}.$$

Again thanks to Lemma A.2 this expression is well-defined if and only if one has the conditions:

$$\mathrm{Im}(P') - 2\mathrm{Re}(\sigma_2) < 0, \quad 2Q - \mathrm{Im}(P') - 2\mathrm{Re}(\sigma_2) > 0.$$

We are free to make a small shift on the contour on integration over σ_2 in order for σ_2 to a small real part which then makes the above inequalities hold. Once the integrals are interchanged, the integration over σ_2 will then act as an inverse Fourier transform on the integral over r'. From here we can compute, using below $\overline{\alpha} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$:

$$\begin{aligned}
&\int_0^\infty d\mu_2 \mu_2^{-a-1} H^{(\alpha_1, Q+iP', \alpha_4)}_{(0,\mu_2,1)} H^{(\alpha_2,\alpha_3,Q-iP')}_{(\mu_2,0,1)} = \frac{4}{\gamma^2} \Gamma(\frac{\alpha_1+\alpha_4+\mathbf{i}P'-Q}{\gamma}) \Gamma(\frac{\alpha_2+\alpha_3-\mathbf{i}P'-Q}{\gamma}) \\
&\times \frac{(2\pi)^{\frac{2Q-\overline{\alpha}}{\gamma}+2} (\frac{2}{\gamma})^{(\frac{\gamma}{2}-\frac{2}{\gamma})(Q-\frac{\overline{\alpha}}{2})-2}}{\Gamma(1-\frac{\gamma^2}{4})^{\frac{2Q-\overline{\alpha}}{\gamma}} \Gamma(\frac{\alpha_1+\alpha_4+\mathbf{i}P'-Q}{\gamma}) \Gamma(\frac{\alpha_2+\alpha_3-\mathbf{i}P'-Q}{\gamma})} \frac{\Gamma_{\frac{\gamma}{2}}(\frac{\alpha_1+\alpha_4-Q-\mathbf{i}P'}{2}) \Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha_1+Q+\mathbf{i}P'-\alpha_4}{2}) \Gamma_{\frac{\gamma}{2}}(Q-\frac{Q+\mathbf{i}P'+\alpha_4-\alpha_1}{2})}{\Gamma_{\frac{\gamma}{2}}(Q)\Gamma_{\frac{\gamma}{2}}(Q-\alpha_1)\Gamma_{\frac{\gamma}{2}}(-\mathbf{i}P')\Gamma_{\frac{\gamma}{2}}(Q-\alpha_4)} \\
&\times \frac{\Gamma_{\frac{\gamma}{2}}(\frac{3}{2}Q-\frac{\alpha_1+\alpha_4+\mathbf{i}P'}{2}) \Gamma_{\frac{\gamma}{2}}(\frac{3}{2}Q-\frac{\alpha_2+\alpha_3-\mathbf{i}P'}{2}) \Gamma_{\frac{\gamma}{2}}(\frac{\alpha_2+Q-\mathbf{i}P'-\alpha_3}{2}) \Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha_2+\alpha_3-Q+\mathbf{i}P'}{2}) \Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha_3+Q-\mathbf{i}P'-\alpha_2}{2})}{\Gamma_{\frac{\gamma}{2}}(Q)\Gamma_{\frac{\gamma}{2}}(Q-\alpha_2)\Gamma_{\frac{\gamma}{2}}(Q-\alpha_3)\Gamma_{\frac{\gamma}{2}}(\mathbf{i}P')} \\
&\times e^{\mathbf{i}\pi\frac{Q}{2}(\alpha_1+\alpha_2+\alpha_3+\alpha_4-2Q)} \times e^{\mathbf{i}\pi\frac{Q}{2}(Q+\mathbf{i}P'-\alpha_2-\alpha_3)} \times \gamma e^{\mathbf{i}\pi\frac{Q}{2}(Q-\mathbf{i}P-\alpha_1-\alpha_2)} e^{\mathbf{i}\pi\frac{Q}{2}(\mathbf{i}P-\mathbf{i}P'+\alpha_2-\alpha_4)} \mathcal{I}_1.
\end{aligned}$$

The last line can by simplified by:

$$e^{\mathbf{i}\pi\frac{Q}{2}(\alpha_1+\alpha_2+\alpha_3+\alpha_4-2Q)} \times e^{\mathbf{i}\pi\frac{Q}{2}(Q+\mathbf{i}P'-\alpha_2-\alpha_3)} \times \gamma e^{\mathbf{i}\pi\frac{Q}{2}(Q-\mathbf{i}P-\alpha_1-\alpha_2)} e^{\mathbf{i}\pi\frac{Q}{2}(\mathbf{i}P-\mathbf{i}P'+\alpha_2-\alpha_4)} = \gamma.$$

Putting everything together this gives the claimed result. □

With this lemma we can now finish the proof of Theorem 3.5.

Proof of Theorem 3.5. We will thus apply our operator $\int_0^\infty d\mu_2 \mu_2^{-a-1}$ to both sides of the second bootstrap statement of Theorem 2.7. By Lemma 3.4 we already know applying this operator to the left hand side will give $\mathcal{G}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(z,P)$ up to a prefactor. For the right hand side by first using Theorem 3.3 we obtain:

$$\int_0^\infty d\mu_2 \mu_2^{-a-1}\left(\int_{\mathbb{R}} H^{(\alpha_1,Q+\mathbf{i}P',\alpha_4)}_{(0,\mu_2,1)} H^{(\alpha_2,\alpha_3,Q-\mathbf{i}P')}_{(\mu_2,0,1)} (1-z)^{\frac{P'^2}{2}} \mathcal{F}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(1-z,P')dP'\right)$$
$$= \int_0^\infty d\mu_2 \mu_2^{-a-1}\left(\int_{\mathbb{R}} H^{(\alpha_1,Q+\mathbf{i}P',\alpha_4)}_{(0,\mu_2,1)} H^{(\alpha_2,\alpha_3,Q-\mathbf{i}P')}_{(\mu_2,0,1)} (1-z)^{\frac{P'^2}{2}} \mathcal{G}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(1-z,P')dP'\right).$$

We then need to justify the exchange of the integral over μ_2 and the one over P'. This is equivalent to showing:

$$\int_0^\infty d\mu_2 \int_{\mathbb{R}} dP' \left|\mu_2^{-a-1} H^{(\alpha_1,Q+\mathbf{i}P',\alpha_4)}_{(0,\mu_2,1)} H^{(\alpha_2,\alpha_3,Q-\mathbf{i}P')}_{(\mu_2,0,1)} (1-z)^{\frac{P'^2}{2}} \mathcal{G}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(1-z,P')\right| < +\infty.$$

Using the bounds (cite), as $\mu_2 \to 0$, the quantity is $\left|H^{(\alpha_1,Q+\mathbf{i}P',\alpha_4)}_{(0,\mu_2,1)} H^{(\alpha_2,\alpha_3,Q-\mathbf{i}P')}_{(\mu_2,0,1)}\right|$ is of constant order in μ_2 and as $\mu_2 \to +\infty$ it will be of order $\mu_2^{\frac{2Q-\alpha_1-\alpha_2-\alpha_3-\alpha_4}{\gamma}}$. In both case the total P dependence grows at most as $Ce^{cP\log|P|}$ and therefore the integral over P' is absolutely converging. For the integration over μ_2, it will thus converge if $\alpha_1+\alpha_2 > Q$ and $\alpha_3+\alpha_4 > Q$. These conditions are indeed satisfied.

This claim should follow from the asymptotics written under Lemma 2.5. Thus by exchanging the integrals and applying Lemma 3.6:

$$\int_0^\infty d\mu_2 \mu_2^{-a-1}\left(\int_{\mathbb{R}} H^{(\alpha_1,Q+\mathbf{i}P',\alpha_4)}_{(0,\mu_2,1)} H^{(\alpha_2,\alpha_3,Q-\mathbf{i}P')}_{(\mu_2,0,1)} (1-z)^{\frac{P'^2}{2}} \mathcal{G}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(1-z,P')dP'\right)$$
$$= \int_{\mathbb{R}} \left(\int_0^\infty d\mu_2 \mu_2^{-a-1} H^{(\alpha_1,Q+\mathbf{i}P',\alpha_4)}_{(0,\mu_2,1)} H^{(\alpha_2,\alpha_3,Q-\mathbf{i}P')}_{(\mu_2,0,1)}\right)(1-z)^{\frac{P'^2}{2}} \mathcal{G}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(1-z,P')dP'$$
$$= \int_{\mathbb{R}} \left(\mathcal{Z}\Gamma(\frac{\alpha_1+\alpha_2+\mathbf{i}P-Q}{\gamma})\Gamma(\frac{\alpha_3+\alpha_4-\mathbf{i}P-Q}{\gamma})\mathcal{M}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(P,P')\right)(1-z)^{\frac{P'^2}{2}} \mathcal{G}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(1-z,P')dP'.$$

Simplifying the prefactors we obtain the result claimed in the theorem. ☐

4. Analytic properties of the fusion kernel and of spherical conformal blocks

4.1. **Identification with the Ponsot-Teschner fusion kernel.** In the physics literature a different expression is given for the fusion kernel. In this subsection we will show that they are equivalent. Consider the following expression of PT for the fusion kernel:

$$\mathcal{M}^{\mathsf{sphere,PT}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(P,P') = \frac{\Gamma_{\frac{\gamma}{2}}(2Q-\beta_1-\beta_2-\beta_3)\Gamma_{\frac{\gamma}{2}}(\beta_2+\beta_3-\beta_1)\Gamma_{\frac{\gamma}{2}}(Q+\beta_2-\beta_1-\beta_3)\Gamma_{\frac{\gamma}{2}}(Q+\beta_3-\beta_2-\beta_1)}{\Gamma_{\frac{\gamma}{2}}(2Q-\sigma_1-\beta_1-\sigma_2)\Gamma_{\frac{\gamma}{2}}(\sigma_1+\sigma_2-\beta_1)\Gamma_{\frac{\gamma}{2}}(Q-\beta_1-\sigma_2+\sigma_1)\Gamma_{\frac{\gamma}{2}}(Q-\beta_1-\sigma_1+\sigma_2)}$$
$$\times \frac{\Gamma_{\frac{\gamma}{2}}(Q-\beta_3-\sigma_1+\sigma_3)\Gamma_{\frac{\gamma}{2}}(\beta_3+\sigma_1+\sigma_3-Q)\Gamma_{\frac{\gamma}{2}}(\sigma_1+\sigma_3-\beta_3)\Gamma_{\frac{\gamma}{2}}(\sigma_3+\beta_3-\sigma_1)}{\Gamma_{\frac{\gamma}{2}}(Q-\beta_2-\sigma_2+\sigma_3)\Gamma_{\frac{\gamma}{2}}(\beta_2+\sigma_2+\sigma_3-Q)\Gamma_{\frac{\gamma}{2}}(\sigma_2+\sigma_3-\beta_2)\Gamma_{\frac{\gamma}{2}}(\sigma_3+\beta_2-\sigma_2)}$$
$$\times \frac{\Gamma_{\frac{\gamma}{2}}(2Q-2\sigma_2)\Gamma_{\frac{\gamma}{2}}(2\sigma_2)}{\Gamma_{\frac{\gamma}{2}}(Q-2\beta_3)\Gamma_{\frac{\gamma}{2}}(2\beta_3-Q)} \frac{1}{\mathbf{i}} \int_{i\mathbb{R}} ds \prod_{i=1}^4 \frac{S_{\frac{\gamma}{2}}(U_i+s)}{S_{\frac{\gamma}{2}}(V_i+s)}$$

where one has:

$$U_1 = \sigma_2+\sigma_1-\beta_1, \qquad V_1 = Q+\sigma_2-\beta_3-\beta_1+\sigma_3,$$
$$U_2 = Q+\sigma_2-\sigma_1-\beta_1, \qquad V_2 = \sigma_2+\beta_3-\beta_1+\sigma_3,$$
$$U_3 = \sigma_2+\beta_2+\sigma_3-Q, \qquad V_3 = 2\sigma_2,$$
$$U_4 = \sigma_2-\beta_2+\sigma_3, \qquad V_4 = Q.$$

and with the following parameter identification:

$$\sigma_1 = \frac{\alpha_1}{2},\ \sigma_2 = \frac{Q+\mathbf{i}P}{2},\ \sigma_3 = \frac{\alpha_4}{2},\ \beta_1 = \frac{\alpha_2}{2},\ \beta_2 = \frac{\alpha_3}{2},\ \beta_3 = \frac{Q+\mathbf{i}P'}{2}.$$

More explicitly, plugging in these parameter we get:

$$U_1 = \frac{Q+\mathbf{i}P}{2} + \frac{\alpha_1}{2} - \frac{\alpha_2}{2}, \qquad V_1 = Q + \frac{Q+\mathbf{i}P}{2} - \frac{Q+\mathbf{i}P'}{2} - \frac{\alpha_2}{2} + \frac{\alpha_4}{2},$$
$$U_2 = Q + \frac{Q+\mathbf{i}P}{2} - \frac{\alpha_1}{2} - \frac{\alpha_2}{2}, \qquad V_2 = \frac{Q+\mathbf{i}P}{2} + \frac{Q+\mathbf{i}P'}{2} - \frac{\alpha_2}{2} + \frac{\alpha_4}{2},$$
$$U_3 = \frac{Q+\mathbf{i}P}{2} + \frac{\alpha_3}{2} + \frac{\alpha_4}{2} - Q, \qquad V_3 = Q + \mathbf{i}P,$$
$$U_4 = \frac{Q+\mathbf{i}P}{2} - \frac{\alpha_3}{2} + \frac{\alpha_4}{2}, \qquad V_4 = Q,$$

and thus:

$$\mathcal{M}^{\mathsf{sphere,PT}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(P,P') = \frac{\Gamma_{\frac{\gamma}{2}}(2Q - \frac{\alpha_2}{2} - \frac{\alpha_3}{2} - \frac{Q+\mathbf{i}P'}{2})\Gamma_{\frac{\gamma}{2}}(\frac{\alpha_3}{2} + \frac{Q+\mathbf{i}P'}{2} - \frac{\alpha_2}{2})\Gamma_{\frac{\gamma}{2}}(Q + \frac{\alpha_3}{2} - \frac{\alpha_2}{2} - \alpha_3)\Gamma_{\frac{\gamma}{2}}(Q + \frac{Q+\mathbf{i}P'}{2} - \frac{\alpha_3}{2} - \frac{\alpha_2}{2})}{\Gamma_{\frac{\gamma}{2}}(2Q - \frac{\alpha_1}{2} - \frac{\alpha_2}{2} - \frac{Q+\mathbf{i}P}{2})\Gamma_{\frac{\gamma}{2}}(\frac{\alpha_1}{2} + \frac{Q+\mathbf{i}P}{2} - \frac{\alpha_2}{2})\Gamma_{\frac{\gamma}{2}}(Q - \frac{\alpha_2}{2} - \frac{Q+\mathbf{i}P}{2} + \frac{\alpha_1}{2})\Gamma_{\frac{\gamma}{2}}(Q - \frac{\alpha_2}{2} - \frac{\alpha_1}{2} + \frac{Q+\mathbf{i}P}{2})}$$
$$\times \frac{\Gamma_{\frac{\gamma}{2}}(Q - \frac{Q+\mathbf{i}P'}{2} - \frac{\alpha_1}{2} + \frac{\alpha_4}{2})\Gamma_{\frac{\gamma}{2}}(\frac{Q+\mathbf{i}P'}{2} + \frac{\alpha_1}{2} + \frac{\alpha_4}{2} - Q)\Gamma_{\frac{\gamma}{2}}(\frac{\alpha_1}{2} + \frac{\alpha_4}{2} - \frac{Q+\mathbf{i}P'}{2})\Gamma_{\frac{\gamma}{2}}(\frac{\alpha_4}{2} + \frac{Q+\mathbf{i}P'}{2} - \frac{\alpha_1}{2})}{\Gamma_{\frac{\gamma}{2}}(Q - \frac{\alpha_3}{2} - \frac{Q+\mathbf{i}P}{2} + \frac{\alpha_4}{2})\Gamma_{\frac{\gamma}{2}}(\frac{\alpha_3}{2} + \frac{Q+\mathbf{i}P}{2} + \frac{\alpha_4}{2} - Q)\Gamma_{\frac{\gamma}{2}}(\frac{Q+\mathbf{i}P}{2} + \frac{\alpha_4}{2} - \frac{\alpha_3}{2})\Gamma_{\frac{\gamma}{2}}(\frac{\alpha_4}{2} + \frac{\alpha_3}{2} - \frac{Q+\mathbf{i}P}{2})}$$
$$\times \frac{\Gamma_{\frac{\gamma}{2}}(Q - \mathbf{i}P)\Gamma_{\frac{\gamma}{2}}(Q + \mathbf{i}P)}{\Gamma_{\frac{\gamma}{2}}(\mathbf{i}P')\Gamma_{\frac{\gamma}{2}}(-\mathbf{i}P')}$$
$$\times \int_{\mathbf{i}\mathbb{R}} \frac{S_{\frac{\gamma}{2}}(\frac{Q+\mathbf{i}P}{2} + \frac{\alpha_1}{2} - \frac{\alpha_2}{2} + r)S_{\frac{\gamma}{2}}(Q + \frac{Q+\mathbf{i}P}{2} - \frac{\alpha_1}{2} - \frac{\alpha_2}{2} + r)S_{\frac{\gamma}{2}}(\frac{Q+\mathbf{i}P}{2} + \frac{\alpha_3}{2} + \frac{\alpha_4}{2} - Q + r)S_{\frac{\gamma}{2}}(\frac{Q+\mathbf{i}P}{2} - \frac{\alpha_3}{2} + \frac{\alpha_4}{2} + r)}{S_{\frac{\gamma}{2}}(Q + \frac{Q+\mathbf{i}P}{2} - \frac{Q+\mathbf{i}P'}{2} - \frac{\alpha_2}{2} + \frac{\alpha_4}{2} + r)S_{\frac{\gamma}{2}}(\frac{Q+\mathbf{i}P}{2} + \frac{Q+\mathbf{i}P'}{2} - \frac{\alpha_2}{2} + \frac{\alpha_4}{2} + r)S_{\frac{\gamma}{2}}(Q + \mathbf{i}P + r)S_{\frac{\gamma}{2}}(Q + r)} dr$$

Proposition 4.1. One has $\mathcal{M}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(P,P') = \frac{1}{2\pi}\mathcal{M}^{\mathsf{sphere,PT}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(P,P')$.

Proof. The proof of Proposition 4.1 is based on Proposition B.3 expressing a symmetry property of hyperbolic Barnes integrals. For $i = 1, \ldots, 8$, define $\widetilde{b}_i = \alpha_i - Q$. For $u \in \mathbb{C}^8$ defined by

$$u = \Big(\frac{1}{4}(-2\mathbf{i}\widetilde{b}_1 + \mathbf{i}\widetilde{b}_2 + \mathbf{i}\widetilde{b}_4 - P + P' + 2\mathbf{i}Q), \frac{1}{4}(2\mathbf{i}\widetilde{b}_1 + \mathbf{i}\widetilde{b}_2 + \mathbf{i}\widetilde{b}_4 - P + P' + 2\mathbf{i}Q),$$
$$\frac{1}{4}(-\mathbf{i}\widetilde{b}_2 + \mathbf{i}\widetilde{b}_4 + P - 3P'), \frac{1}{4}(\mathbf{i}\widetilde{b}_2 - \mathbf{i}\widetilde{b}_4 - P - P'), \frac{1}{4}(-\mathbf{i}\widetilde{b}_2 + \mathbf{i}\widetilde{b}_4 + P + P'), \frac{1}{4}(\mathbf{i}\widetilde{b}_2 - \mathbf{i}\widetilde{b}_4 + 3P - P'),$$
$$\frac{1}{4}(-\mathbf{i}\widetilde{b}_2 - 2\mathbf{i}\widetilde{b}_3 - \mathbf{i}\widetilde{b}_4 - P + P' - 2\mathbf{i}Q), \frac{1}{4}(-\mathbf{i}\widetilde{b}_2 + 2\mathbf{i}\widetilde{b}_3 - \mathbf{i}\widetilde{b}_4 - P + P' + 2\mathbf{i}Q)\Big),$$

we find that $u' = w \cdot u$ is given by

$$u' = \Big(\frac{1}{2}(-\mathbf{i}\widetilde{b}_1 + P' + \mathbf{i}Q), \frac{1}{2}(\mathbf{i}\widetilde{b}_1 + P' + \mathbf{i}Q), \frac{1}{2}(-\mathbf{i}\widetilde{b}_2 + P - P'), \frac{1}{2}(-\mathbf{i}\widetilde{b}_4),$$
$$\frac{1}{2}(\mathbf{i}\widetilde{b}_4), \frac{1}{2}(\mathbf{i}\widetilde{b}_2 + P - P'), \frac{1}{2}(-\mathbf{i}\widetilde{b}_3 - P + \mathbf{i}Q), \frac{1}{2}(\mathbf{i}\widetilde{b}_3 - P + \mathbf{i}Q)\Big).$$

Define the integral expression

$$\mathcal{I} := \int_{\mathbf{i}\mathbb{R}} \frac{S_{\frac{\gamma}{2}}(\frac{Q+\mathbf{i}P}{2} + \frac{\alpha_1}{2} - \frac{\alpha_2}{2} + r)S_{\frac{\gamma}{2}}(Q + \frac{Q+\mathbf{i}P}{2} - \frac{\alpha_1}{2} - \frac{\alpha_2}{2} + r)S_{\frac{\gamma}{2}}(\frac{Q+\mathbf{i}P}{2} + \frac{\alpha_3}{2} + \frac{\alpha_4}{2} - Q + r)S_{\frac{\gamma}{2}}(\frac{Q+\mathbf{i}P}{2} - \frac{\alpha_3}{2} + \frac{\alpha_4}{2} + r)}{S_{\frac{\gamma}{2}}(Q + \frac{Q+\mathbf{i}P}{2} - \frac{Q+\mathbf{i}P'}{2} - \frac{\alpha_2}{2} + \frac{\alpha_4}{2} + r)S_{\frac{\gamma}{2}}(\frac{Q+\mathbf{i}P}{2} + \frac{Q+\mathbf{i}P'}{2} - \frac{\alpha_2}{2} + \frac{\alpha_4}{2} + r)S_{\frac{\gamma}{2}}(Q + \mathbf{i}P + r)S_{\frac{\gamma}{2}}(Q + r)} dr.$$

Shifting the integration contour of $\mathcal{I}_1$ by $-\frac{1}{2}\beta_4$ and the integration contour of $\mathcal{I}$ by $\frac{1}{4}\widetilde{b}_2 - \frac{1}{4}\widetilde{b}_4 - \frac{1}{2}Q + \frac{1}{4}\mathbf{i}P' - \frac{1}{4}\mathbf{i}P$ and applying Proposition B.3, we obtain that

$$\mathcal{I}_1 = \mathcal{I} \cdot S_{\frac{\gamma}{2}}((\pm\widetilde{b}_1 + \widetilde{b}_4 + \mathbf{i}P' + Q)/2)S_{\frac{\gamma}{2}}((\pm\widetilde{b}_1 + \widetilde{b}_2 + \mathbf{i}P + Q)/2)$$
$$S_{\frac{\gamma}{2}}((-\widetilde{b}_2 \pm \widetilde{b}_3 - \mathbf{i}P' + Q)/2)S_{\frac{\gamma}{2}}((\pm\widetilde{b}_3 - \widetilde{b}_4 - \mathbf{i}P + Q)/2). \tag{4.1}$$

Simplifying the prefactors of the integrals in $\mathcal{M}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(P,P')$ and $\mathcal{M}^{\mathsf{sphere,PT}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(P,P')$, we find that (4.1) implies the claim. $\square$

4.2. **Analytic properties of the fusion kernel.** In order to prove analytic properties of the 4-point sphere conformal block, it is required to fully understand the analytic properties of the fusion kernel. Consider:

$$P_{m,n} = \frac{\mathbf{i}\gamma m}{2} + \frac{2\mathbf{i}n}{\gamma}.$$

We will prove the following proposition.

Proposition 4.2. Fix $P' \in \mathbb{R}$. Let also α_i satisfy $\alpha_i < Q$, $\alpha_1 + \alpha_2 > Q$, $\alpha_3 + \alpha_4 > Q$, $\alpha_1 + \alpha_4 > Q$, $\alpha_2 + \alpha_3 > Q$. Then the function $P \mapsto \mathcal{M}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(P, P')$ is meromorphic on $\mathbb{C}$ with simple poles at $P = \pm P_{m,n}$, for $m, n \geq 1$. Furthermore the residues at the poles are given by

$$\mathrm{Res}_{P=P_{m,n}} \mathcal{M}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(P, P') = c(m, n, P)\mathcal{M}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(P_{-m,n}, P'), \tag{4.2}$$

where we have introduced

$$c(m, n, P) := -\mathbf{i}(-1)^{mn} \left(\prod_{(j,k)\in S_{m,n}} \frac{1}{\frac{j\gamma}{2} + \frac{2k}{\gamma}} \right) \prod_{j=1}^{4} \prod_{i=0}^{m-1} \prod_{k=0}^{n-1} \left(T_j - \frac{\gamma m}{4} + \frac{\gamma i}{2} - \frac{n}{\gamma} + \frac{2k}{\gamma} \right), \tag{4.3}$$

and used the notations $S_{m,n} := \{(j, k) \in \mathbb{N}^2, -m + 1 \leq j \leq m, -n + 1 \leq k \leq n, (j, k) \neq (0, 0)\}$ and:

$$T_1 = \frac{\alpha_1 + \alpha_2 - Q}{2}, \quad T_2 = \frac{\alpha_2 - \alpha_1 + Q}{2}, \quad T_3 = \frac{\alpha_3 - \alpha_4 + Q}{2}, \quad T_4 = \frac{3Q - \alpha_3 - \alpha_4}{2}.$$

Proof. The function $\mathcal{M}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(P, P')$ contains the product $\Gamma_{\frac{\gamma}{2}}(Q - \mathbf{i}P)\Gamma_{\frac{\gamma}{2}}(Q + \mathbf{i}P)$, which has poles at the desired locations, $P = \pm P_{m,n}$, $m, n \geq 1$. Establishing the proposition will thus contain two steps: i) show that the function $\mathcal{M}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(P, P')$ is analytic in P on $\mathbb{C} \setminus \cup_{m,n\geq 1}\{P_{m,n}, -P_{m,n}\}$, ii) establish the value of the residue at $P = \pm P_{m,n}$.
Step 1. Let us introduce the following function of P:

$$\begin{aligned} f(P) := &\frac{1}{\Gamma_{\frac{\gamma}{2}}(\frac{\alpha_1+\alpha_2-Q+\mathbf{i}P}{2})\Gamma_{\frac{\gamma}{2}}(\frac{\alpha_1+Q+\mathbf{i}P-\alpha_2}{2})\Gamma_{\frac{\gamma}{2}}(\frac{\alpha_2+Q+\mathbf{i}P-\alpha_1}{2})\Gamma_{\frac{\gamma}{2}}(Q - \frac{\alpha_1+\alpha_2-Q-\mathbf{i}P}{2})} \\ &\times \frac{1}{\Gamma_{\frac{\gamma}{2}}(\frac{-Q+\alpha_3+\alpha_4-\mathbf{i}P}{2})\Gamma_{\frac{\gamma}{2}}(\frac{\alpha_3+Q-\mathbf{i}P-\alpha_4}{2})\Gamma_{\frac{\gamma}{2}}(\frac{\alpha_4+Q-\mathbf{i}P-\alpha_3}{2})\Gamma_{\frac{\gamma}{2}}(Q - \frac{\alpha_3+\alpha_4-Q+\mathbf{i}P}{2})} \mathcal{I}_1. \end{aligned} \tag{4.4}$$

The goal of thus to show that $f(P)$ is an analytic function of P on $\mathbb{C}$. For this we will use the following lemma giving the set of poles of $\mathcal{I}_1$.

Lemma 4.3. The function $\mathcal{I}_1$ is a meromorphic function on $\mathbb{C}^6$ of all its parameters $\alpha_1, \alpha_2, \alpha_3, \alpha_4, P, P'$ and has poles when $\xi = \frac{n\gamma}{2} + \frac{2m}{\gamma}$, for $m, n \geq 1$, where ξ can be equal to any of the following:

$$\begin{gathered} \frac{\alpha_1 - \alpha_4 - \mathbf{i}P' - Q}{2}, \quad \frac{Q - \alpha_1 - \alpha_4 - \mathbf{i}P'}{2}, \quad \frac{\alpha_3 - \alpha_4 + \mathbf{i}P - Q}{2}, \quad \frac{Q - \alpha_3 - \alpha_4 + \mathbf{i}P}{2}, \\ \frac{\alpha_1 + \alpha_4 - \mathbf{i}P' - 3Q}{2}, \quad \frac{-Q - \alpha_1 + \alpha_4 - \mathbf{i}P'}{2}, \quad \frac{\alpha_3 + \alpha_4 + \mathbf{i}P - 3Q}{2}, \quad \frac{-Q - \alpha_3 + \alpha_4 + \mathbf{i}P}{2}, \\ \frac{\alpha_1 - \alpha_2 - \mathbf{i}P - Q}{2}, \quad \frac{Q - \alpha_1 - \alpha_2 - \mathbf{i}P}{2}, \quad \frac{\alpha_3 - \alpha_2 + \mathbf{i}P' - Q}{2}, \quad \frac{Q - \alpha_3 - \alpha_2 + \mathbf{i}P'}{2}, \\ \frac{\alpha_1 + \alpha_2 - \mathbf{i}P - 3Q}{2}, \quad \frac{-Q - \alpha_1 + \alpha_2 - \mathbf{i}P}{2}, \quad \frac{\alpha_3 + \alpha_2 + \mathbf{i}P' - 3Q}{2}, \quad \frac{-Q - \alpha_3 + \alpha_2 + \mathbf{i}P'}{2}. \end{gathered}$$

Proof. Let us rewrite the expression of $\mathcal{I}_1$:

$$\mathcal{I}_1 = \int_{\mathcal{C}} \frac{S_{\frac{\gamma}{2}}(\frac{Q+\mathbf{i}P'}{2} + \frac{\alpha_4}{2} - \frac{\alpha_1}{2} + r)S_{\frac{\gamma}{2}}(\frac{\alpha_1+\alpha_4+\mathbf{i}P'}{2} - \frac{Q}{2} + r)S_{\frac{\gamma}{2}}(\frac{Q}{2} - \frac{\alpha_3}{2} + r - \frac{\mathbf{i}P}{2} + \frac{\alpha_4}{2})S_{\frac{\gamma}{2}}(\frac{\alpha_3}{2} - \frac{Q}{2} + r - \frac{\mathbf{i}P}{2} + \frac{\alpha_4}{2})}{S_{\frac{\gamma}{2}}(Q + r)S_{\frac{\gamma}{2}}(\alpha_4 + r)S_{\frac{\gamma}{2}}(Q + r - \frac{\mathbf{i}P}{2} + \frac{\mathbf{i}P'}{2} + \frac{\alpha_4}{2} - \frac{\alpha_2}{2})S_{\frac{\gamma}{2}}(\frac{\alpha_2}{2} + r - \frac{\mathbf{i}P}{2} + \frac{\mathbf{i}P'}{2} + \frac{\alpha_4}{2})} \frac{dr}{i}.$$

Recall also that the function $S_{\frac{\gamma}{2}}(x)$ has poles at $x = -\frac{\gamma n}{2} - \frac{2m}{\gamma}$ and zeros at $x = Q + \frac{\gamma n}{2} + \frac{2m}{\gamma}$ for any $m, n \geq 0$. As a function or r the integrand in the expression of $\mathcal{I}_1$ has poles at the following locations, for any $m, n \geq 0$:

$$\frac{\alpha_1 - \alpha_4 - \mathbf{i}P' - Q}{2} - \frac{\gamma n}{2} - \frac{2m}{\gamma}, \quad \frac{Q - \alpha_1 - \alpha_4 - \mathbf{i}P'}{2} - \frac{\gamma n}{2} - \frac{2m}{\gamma},$$
$$\frac{\alpha_3 - \alpha_4 + \mathbf{i}P - Q}{2} - \frac{\gamma n}{2} - \frac{2m}{\gamma}, \quad \frac{Q - \alpha_3 - \alpha_4 + \mathbf{i}P}{2} - \frac{\gamma n}{2} - \frac{2m}{\gamma},$$
$$\frac{\mathbf{i}P - \mathbf{i}P' + \alpha_2 - \alpha_4}{2} + \frac{\gamma n}{2} + \frac{2m}{\gamma}, \quad Q + \frac{\mathbf{i}P - \mathbf{i}P' - \alpha_2 - \alpha_4}{2} + \frac{\gamma n}{2} + \frac{2m}{\gamma},$$
$$\frac{\gamma n}{2} + \frac{2m}{\gamma}, \quad Q - \alpha_4 + \frac{\gamma n}{2} + \frac{2m}{\gamma}.$$

We thus have four lattices of poles extending in the positive real direction and four lattices of poles extending in the negative real direction. The function $\mathcal{I}_1$ has then a pole when the parameters are such that the integrand of $\mathcal{I}_1$ as a function of r has a pole of a lattice extending to the right collapse with a pole of the lattice extending to the left. See the proof of [RZ21] for more details on how these poles appear. By writing out all the possible combinations, this gives the claim list of poles of the lemma. □

Using this lemma it is now easy to conclude that $f(P)$ is an analytic function of P on all of $\mathbb{C}$, since the poles coming from the integral $\mathcal{I}_1$ are cancelled by the poles of the double gamma functions in the denominator of the prefactor in front of $\mathcal{I}_1$. This completes step one.

Step 2. In this second step we will establish the claimed formula on the residues at $P = \pm P_{m,n}$. The non trivial step is to relate $\mathcal{I}_1$ at $P = P_{m,n}$ and $P_{-m,n}$. For this we will need the following cyclic permutation identity which is proved in Appendix B.

(4.5)

$$\int_{\mathcal{C}} \frac{S_{\frac{\gamma}{2}}(-\frac{\beta_2}{2} + \sigma_2 + \sigma_3 + r) S_{\frac{\gamma}{2}}(Q - \frac{\beta_2}{2} + \sigma_3 - \sigma_2 + r) S_{\frac{\gamma}{2}}(\frac{\beta_3}{2} + \sigma_3 - \sigma_1 + r) S_{\frac{\gamma}{2}}(Q - \frac{\beta_3}{2} + \sigma_3 - \sigma_1 + r)}{S_{\frac{\gamma}{2}}(Q + \frac{\beta_1}{2} - \frac{\beta_2}{2} + \sigma_3 - \sigma_1 + r) S_{\frac{\gamma}{2}}(2Q - \frac{\beta_1}{2} - \frac{\beta_2}{2} + \sigma_3 - \sigma_1 + r) S_{\frac{\gamma}{2}}(2\sigma_3 + r) S_{\frac{\gamma}{2}}(Q + r)} \frac{dr}{i}$$
$$= \frac{\Gamma_{\frac{\gamma}{2}}(\frac{\beta_3 + \beta_2 - \beta_1}{2}) \Gamma_{\frac{\gamma}{2}}(Q - \frac{\beta_3 + \beta_1 - \beta_2}{2}) S_{\frac{\gamma}{2}}(\frac{\beta_1}{2} + \sigma_1 - \sigma_2) S_{\frac{\gamma}{2}}(\frac{\beta_1}{2} + \sigma_1 + \sigma_2 - Q) S_{\frac{\gamma}{2}}(\frac{\beta_3}{2} - \sigma_1 - \sigma_3 + Q)}{\Gamma_{\frac{\gamma}{2}}(\frac{\beta_1 + \beta_3 - \beta_2}{2}) \Gamma_{\frac{\gamma}{2}}(Q - \frac{\beta_2 + \beta_3 - \beta_1}{2}) S_{\frac{\gamma}{2}}(\frac{\beta_2}{2} + \sigma_2 - \sigma_3) S_{\frac{\gamma}{2}}(\frac{\beta_3}{2} + \sigma_3 + \sigma_1 - Q) S_{\frac{\gamma}{2}}(\frac{\beta_2}{2} - \sigma_3 - \sigma_2 + Q)}$$
$$\times \int_{\mathcal{C}} \frac{S_{\frac{\gamma}{2}}(-\frac{\beta_1}{2} + \sigma_1 + \sigma_2 + r) S_{\frac{\gamma}{2}}(Q - \frac{\beta_1}{2} + \sigma_2 - \sigma_1 + r) S_{\frac{\gamma}{2}}(\frac{\beta_2}{2} + \sigma_2 - \sigma_3 + r) S_{\frac{\gamma}{2}}(Q - \frac{\beta_2}{2} + \sigma_2 - \sigma_3 + r)}{S_{\frac{\gamma}{2}}(Q + \frac{\beta_3}{2} - \frac{\beta_1}{2} + \sigma_2 - \sigma_3 + r) S_{\frac{\gamma}{2}}(2Q - \frac{\beta_3}{2} - \frac{\beta_1}{2} + \sigma_2 - \sigma_3 + r) S_{\frac{\gamma}{2}}(2\sigma_2 + r) S_{\frac{\gamma}{2}}(Q + r)} \frac{dr}{i}$$

In this equality let us plug in the parameter substitution:

$$\sigma_1 = \frac{Q - \mathbf{i}P'}{2}, \quad \sigma_2 = \frac{\alpha_3}{2}, \quad \sigma_3 = \frac{\alpha_4}{2}, \quad \beta_1 = \alpha_2, \quad \beta_2 = Q + \mathbf{i}P, \quad \beta_3 = \alpha_1.$$

Under this choice, the integral of the first line becomes $\mathcal{I}_1$ and the integral of the third line becomes:

$$\mathcal{J}_1(P) = \int_{\mathcal{C}} \frac{S_{\frac{\gamma}{2}}(-\frac{\alpha_2}{2} + \frac{Q - \mathbf{i}P'}{2} + \frac{\alpha_3}{2} + r) S_{\frac{\gamma}{2}}(\frac{Q}{2} - \frac{\alpha_2}{2} + \frac{\alpha_3}{2} + \frac{\mathbf{i}P'}{2} + r) S_{\frac{\gamma}{2}}(\frac{Q + \mathbf{i}P}{2} + \frac{\alpha_3}{2} - \frac{\alpha_4}{2} + r) S_{\frac{\gamma}{2}}(\frac{Q}{2} - \frac{\mathbf{i}P}{2} + \frac{\alpha_3}{2} - \frac{\alpha_4}{2} + r)}{S_{\frac{\gamma}{2}}(Q + \frac{\alpha_1}{2} - \frac{\alpha_2}{2} + \frac{\alpha_3}{2} - \frac{\alpha_4}{2} + r) S_{\frac{\gamma}{2}}(2Q - \frac{\alpha_1}{2} - \frac{\alpha_2}{2} + \frac{\alpha_3}{2} - \frac{\alpha_4}{2} + r) S_{\frac{\gamma}{2}}(\alpha_3 + r) S_{\frac{\gamma}{2}}(Q + r)} \frac{dr}{i}.$$

Under this form it is easy to relate $\mathcal{J}_1(P_{m,n})$ and $\mathcal{J}_1(P_{-m,n})$. Using (A.5) it is immediate to derive the following identity:

$$\begin{aligned} S_{\frac{\gamma}{2}}(T + \frac{P_{-m,n}}{2}) S_{\frac{\gamma}{2}}(T - \frac{P_{-m,n}}{2}) &= S_{\frac{\gamma}{2}}(T - \frac{\gamma m}{4} + \frac{n}{\gamma}) S_{\frac{\gamma}{2}}(T + \frac{\gamma m}{4} - \frac{n}{\gamma}) \\ &= \frac{\prod_{i=1}^m 2\sin(\pi \frac{\gamma}{2}(T + \frac{\gamma}{4}(m - 2i) - \frac{n}{\gamma}))}{\prod_{i=1}^m 2\sin(\pi \frac{\gamma}{2}(T + \frac{\gamma}{4}(m - 2i) + \frac{n}{\gamma}))} S_{\frac{\gamma}{2}}(T + \frac{\gamma m}{4} + \frac{n}{\gamma}) S_{\frac{\gamma}{2}}(T - \frac{\gamma m}{4} - \frac{n}{\gamma}) \\ &= (-1)^{mn} S_{\frac{\gamma}{2}}(T + \frac{P_{m,n}}{2}) S_{\frac{\gamma}{2}}(T - \frac{P_{m,n}}{2}). \end{aligned}$$

Then setting $T = \frac{Q + \alpha_3 - \alpha_4}{2} + r$ immediately gives $\mathcal{J}_1(P_{m,n}) = (-1)^{mn} \mathcal{J}_1(P_{-m,n})$. Finally we need to take a look at the P dependence of all $\Gamma_{\frac{\gamma}{2}}$ and $S_{\frac{\gamma}{2}}$ functions in front of $\mathcal{J}_1(P)$. Combing the prefactors in front of

$\mathcal{I}_1$ in (4.4) and the expression in the second line of (4.5) and obtains the function:

$$\begin{aligned}
&\frac{\Gamma_{\frac{\gamma}{2}}(Q-\mathbf{i}P)\Gamma_{\frac{\gamma}{2}}(Q+\mathbf{i}P)}{\Gamma_{\frac{\gamma}{2}}(\frac{\alpha_1+\alpha_2-Q+\mathbf{i}P}{2})\Gamma_{\frac{\gamma}{2}}(\frac{\alpha_1+Q+\mathbf{i}P-\alpha_2}{2})\Gamma_{\frac{\gamma}{2}}(\frac{\alpha_2+Q+\mathbf{i}P-\alpha_1}{2})\Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha_1+\alpha_2-Q-\mathbf{i}P}{2})}\\
&\times\frac{1}{\Gamma_{\frac{\gamma}{2}}(\frac{-Q+\alpha_3+\alpha_4-\mathbf{i}P}{2})\Gamma_{\frac{\gamma}{2}}(\frac{\alpha_3+Q-\mathbf{i}P-\alpha_4}{2})\Gamma_{\frac{\gamma}{2}}(\frac{\alpha_4+Q-\mathbf{i}P-\alpha_3}{2})\Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha_3+\alpha_4-Q+\mathbf{i}P}{2})}\\
&\times\frac{\Gamma_{\frac{\gamma}{2}}(\frac{\alpha_1+Q+\mathbf{i}P-\alpha_2}{2})\Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha_1+\alpha_2-Q-\mathbf{i}P}{2})}{\Gamma_{\frac{\gamma}{2}}(\frac{\alpha_2+\alpha_1-Q-\mathbf{i}P}{2})\Gamma_{\frac{\gamma}{2}}(Q-\frac{Q+\mathbf{i}P+\alpha_1-\alpha_2}{2})S_{\frac{\gamma}{2}}(\frac{Q+\mathbf{i}P}{2}+\frac{\alpha_3}{2}-\frac{\alpha_4}{2})S_{\frac{\gamma}{2}}(\frac{Q+\mathbf{i}P}{2}-\frac{\alpha_4}{2}-\frac{\alpha_3}{2}+Q)}\\
&=\frac{\Gamma_{\frac{\gamma}{2}}(Q\pm\mathbf{i}P)}{\Gamma_{\frac{\gamma}{2}}(\frac{\alpha_1+\alpha_2-Q\pm\mathbf{i}P}{2})\Gamma_{\frac{\gamma}{2}}(\frac{\alpha_2+Q\pm\mathbf{i}P-\alpha_1}{2})\Gamma_{\frac{\gamma}{2}}(\frac{\alpha_3+Q\pm\mathbf{i}P-\alpha_4}{2})\Gamma_{\frac{\gamma}{2}}(\frac{3Q-\alpha_3-\alpha_4\pm\mathbf{i}P}{2})}.
\end{aligned}$$

To establish an identity of the type $\mathrm{Res}_{P=P_{m,n}}\mathcal{M}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(P,P')=c(m,n,P)\mathcal{M}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(P_{-m,n},P')$, we first compute the residue of $\Gamma_{\frac{\gamma}{2}}(Q\pm\mathbf{i}P)$. We write out:

$$\begin{aligned}
&\Gamma_{\frac{\gamma}{2}}(Q+\mathbf{i}P)\Gamma_{\frac{\gamma}{2}}(Q-\mathbf{i}P)\\
&=\left(\prod_{j=0}^{2m-1}\left(\frac{2}{\gamma}\right)^{\frac{\gamma}{2}(2\mathbf{i}P+(2j+1)\frac{\gamma}{2})}\frac{\Gamma(\frac{\gamma}{2}(Q+\mathbf{i}P+j\frac{\gamma}{2}))}{\Gamma(\frac{\gamma}{2}(Q-\mathbf{i}P-(j+1)\frac{\gamma}{2}))}\right)\Gamma_{\frac{\gamma}{2}}(Q+\mathbf{i}P+\gamma m)\Gamma_{\frac{\gamma}{2}}(Q-\mathbf{i}P-\gamma m).
\end{aligned}$$

Record the identity:

$$\begin{aligned}
\lim_{P\to P_{m,n}}(P-P_{m,n})\frac{\Gamma(\frac{\gamma}{2}(Q+\mathbf{i}P+(m-1)\frac{\gamma}{2}))}{\Gamma(\frac{\gamma}{2}(Q-\mathbf{i}P-(m+1)\frac{\gamma}{2}))}&=\lim_{P\to\frac{\mathbf{i}\gamma m}{2}}(P-\frac{\mathbf{i}\gamma m}{2})\frac{\Gamma(\frac{\gamma}{2}(\frac{2}{\gamma}-\frac{2n}{\gamma}+\mathbf{i}P+\frac{m\gamma}{2}))}{\Gamma(\frac{\gamma}{2}(\frac{2}{\gamma}+\frac{2n}{\gamma}-\mathbf{i}P-\frac{m\gamma}{2}))}\\
&=\frac{1}{n!}\lim_{P\to\frac{\mathbf{i}\gamma m}{2}}(P-\frac{\mathbf{i}\gamma m}{2})\prod_{k=1}^{n}\frac{1}{k-n+\frac{\gamma}{2}(\mathbf{i}P+\frac{m\gamma}{2})}\\
&=-\frac{2\mathbf{i}}{\gamma}\frac{(-1)^{n-1}}{n!(n-1)!}.
\end{aligned}$$

This implies:

$$\begin{aligned}
&\lim_{P\to P_{m,n}}(P-P_{m,n})\left(\prod_{j=0}^{2m-1}\left(\frac{2}{\gamma}\right)^{\frac{\gamma}{2}(2\mathbf{i}P+(2j+1)\frac{\gamma}{2})}\frac{\Gamma(\frac{\gamma}{2}(Q+\mathbf{i}P+j\frac{\gamma}{2}))}{\Gamma(\frac{\gamma}{2}(Q-\mathbf{i}P-(j+1)\frac{\gamma}{2}))}\right)\\
&=\left(\frac{\gamma}{2}\right)^{4nm}\left(\prod_{j=-m,j\neq-1}^{m-1}\frac{\Gamma(\frac{\gamma}{2}(Q-\frac{2n}{\gamma}+j\frac{\gamma}{2}))}{\Gamma(\frac{\gamma}{2}(Q+\frac{2n}{\gamma}+j\frac{\gamma}{2}))}\right)\lim_{P\to P_{m,n}}(P-P_{m,n})\frac{\Gamma(\frac{\gamma}{2}(Q+\mathbf{i}P+(m-1)\frac{\gamma}{2}))}{\Gamma(\frac{\gamma}{2}(Q-\mathbf{i}P-(m+1)\frac{\gamma}{2}))}\\
&=\left(\frac{\gamma}{2}\right)^{4nm}\left(\prod_{j=-m,j\neq-1}^{m-1}\prod_{k=-n}^{n-1}\frac{1}{\frac{\gamma}{2}(Q+\frac{j\gamma}{2})+k}\right)\left(-\frac{2\mathbf{i}}{\gamma}\frac{(-1)^{n-1}}{n!(n-1)!}\right)=-\mathbf{i}\prod_{(j,k)\in S_{m,n}}\frac{1}{\frac{j\gamma}{2}+\frac{2k}{\gamma}},
\end{aligned}$$

where we have used the notation $S_{m,n}:=\{(j,k)\in\mathbb{N}^2,-m+1\leq j\leq m,-n+1\leq k\leq n,(j,k)\neq(0,0)\}$. From here we conclude that:

$$\mathrm{Res}_{P=P_{m,n}}\Gamma_{\frac{\gamma}{2}}(Q+\mathbf{i}P)\Gamma_{\frac{\gamma}{2}}(Q-\mathbf{i}P)=-\mathbf{i}\prod_{(j,k)\in S_{m,n}}\frac{1}{\frac{j\gamma}{2}+\frac{2k}{\gamma}}\Gamma_{\frac{\gamma}{2}}(Q+\mathbf{i}P_{-m,n})\Gamma_{\frac{\gamma}{2}}(Q-\mathbf{i}P_{-m,n}).$$

Next using (A.3) one can easily derive:

$$\begin{aligned}
&\Gamma_{\frac{\gamma}{2}}(T+\frac{\mathbf{i}P_{m,n}}{2})\Gamma_{\frac{\gamma}{2}}(T-\frac{\mathbf{i}P_{m,n}}{2})\\
&=\prod_{i=0}^{m-1}\left(\frac{\Gamma(\frac{\gamma}{2}(T-\frac{n}{\gamma}-\frac{\gamma m}{4}+\frac{\gamma i}{2}))}{\Gamma(\frac{\gamma}{2}(T+\frac{n}{\gamma}-\frac{\gamma m}{4}+\frac{\gamma i}{2}))}\frac{(\frac{\gamma}{2})^{\frac{\gamma}{2}(\frac{n}{\gamma})}}{(\frac{\gamma}{2})^{\frac{\gamma}{2}(-\frac{n}{\gamma})}}\right)\Gamma_{\frac{\gamma}{2}}(T+\frac{\mathbf{i}P_{-m,n}}{2})\Gamma_{\frac{\gamma}{2}}(T-\frac{\mathbf{i}P_{-m,n}}{2})\\
&=\prod_{i=0}^{m-1}\prod_{k=0}^{n-1}\left(\frac{1}{T-\frac{\gamma m}{4}+\frac{\gamma i}{2}-\frac{n}{\gamma}+\frac{2k}{\gamma}}\right)\Gamma_{\frac{\gamma}{2}}(T+\frac{\mathbf{i}P_{-m,n}}{2})\Gamma_{\frac{\gamma}{2}}(T-\frac{\mathbf{i}P_{-m,n}}{2}).
\end{aligned}$$

Therefore we arrive at:

$$\Gamma_{\frac{\gamma}{2}}(\frac{\alpha_1+\alpha_2-Q\pm\mathbf{i}P_{m,n}}{2})\Gamma_{\frac{\gamma}{2}}(\frac{\alpha_2+Q\pm\mathbf{i}P_{m,n}-\alpha_1}{2})\Gamma_{\frac{\gamma}{2}}(\frac{\alpha_3+Q\pm\mathbf{i}P_{m,n}-\alpha_4}{2})\Gamma_{\frac{\gamma}{2}}(\frac{3Q-\alpha_3-\alpha_4\pm\mathbf{i}P_{m,n}}{2})$$
$$=\prod_{j=1}^{4}\prod_{i=0}^{m-1}\prod_{k=0}^{n-1}\left(\frac{1}{T_j-\frac{\gamma m}{4}+\frac{\gamma i}{2}-\frac{n}{\gamma}+\frac{2k}{\gamma}}\right)$$
$$\times\Gamma_{\frac{\gamma}{2}}(\frac{\alpha_1+\alpha_2-Q\pm\mathbf{i}P_{-m,n}}{2})\Gamma_{\frac{\gamma}{2}}(\frac{\alpha_2+Q\pm\mathbf{i}P_{-m,n}-\alpha_1}{2})\Gamma_{\frac{\gamma}{2}}(\frac{\alpha_3+Q\pm\mathbf{i}P_{-m,n}-\alpha_4}{2})\Gamma_{\frac{\gamma}{2}}(\frac{3Q-\alpha_3-\alpha_4\pm\mathbf{i}P_{-m,n}}{2}),$$

where we have used the notation:

$$T_1=\frac{\alpha_1+\alpha_2-Q}{2},\quad T_2=\frac{\alpha_2-\alpha_1+Q}{2},\quad T_3=\frac{\alpha_3-\alpha_4+Q}{2},\quad T_4=\frac{3Q-\alpha_3-\alpha_4}{2}.$$

Putting all the above steps together we arrive at the claimed expression of $c(m,n,P)$:

$$c(m,n,P):=-\mathbf{i}(-1)^{mn}\left(\prod_{(j,k)\in S_{m,n}}\frac{1}{\frac{j\gamma}{2}+\frac{2k}{\gamma}}\right)\prod_{j=1}^{4}\prod_{i=0}^{m-1}\prod_{k=0}^{n-1}\left(T_j-\frac{\gamma m}{4}+\frac{\gamma i}{2}-\frac{n}{\gamma}+\frac{2k}{\gamma}\right). \tag{4.6}$$

This completes the proof of the proposition.

□

4.3. **Analytic properties of the spherical conformal block.** Taking as an input of the results of the previous section, we are now able to finish the proof of Theorem 1.2.

Proof of Theorem 1.2. Recall again:

$$P_{m,n}=\frac{\mathbf{i}\gamma m}{2}+\frac{2\mathbf{i}n}{\gamma}.$$

We will prove the four claims of the theorem in order. First for claim (a), fix $z\in\mathbb{D}$ and $P\neq P_{m,n}$. We have established the fusion transformation in Theorem 3.5 in the parameter range range where $\mathcal{G}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(z,P)$ is well-defined. By using the almost sure equality of $\mathcal{G}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(z,P)$ and $\mathcal{F}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(z,P)$ proved in Theorem 3.3 we can further write:

$$z^{\frac{1}{2}P^2}\mathcal{G}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(z,P)=C(z)\int_{\mathbb{R}}\mathcal{M}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(P,P')(1-z)^{\frac{1}{2}(P')^2}\mathcal{F}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(1-z,P')dP'. \tag{4.7}$$

Now here we will simply take the integral on the right hand side as the definition of the conformal block for any z and P. For almost every P', the integrand of the integral on the right side is jointly analytic in P outside of the poles of $\mathcal{M}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(P,P')$, which precisely corresponds to choosing $P\neq P_{m,n}$, and analytic in the z variable thanks to the following lemmas. To say that these properties imply the block on the left hand side has the same properties we need to show the following integral is finite:

$$\int_{\mathbb{R}}\left|\mathcal{M}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(P,P')(1-z)^{\frac{1}{2}(P')^2}\mathcal{F}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(1-z,P')\right|dP'<\infty.$$

For this purpose we give the following lemmas proved in [GKRV20]. Recall the coefficients β_n of the series expansion of the conformal blocks given in introduction by (1.6).

Lemma 4.4. Fix parameters $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ and $P\in\mathbb{R}$. Then one has:

$$|\beta_n(\Delta_{Q+iP},\Delta_{\alpha_1},\Delta_{\alpha_2},\Delta_{\alpha_3},\Delta_{\alpha_4})|\leq\frac{1}{2}(\beta_n(\Delta_{Q+iP},\Delta_{\alpha_1},\Delta_{\alpha_2},\Delta_{\alpha_2},\Delta_{\alpha_1})+\beta_n(\Delta_{Q+iP},\Delta_{\alpha_4},\Delta_{\alpha_3},\Delta_{\alpha_3},\Delta_{\alpha_4})).$$

Proof. Since the matrix $(F^{-1}_{Q+iP}(\nu,\nu'))_{|\nu|,|\nu'|=n}$ is positive definite, by Cauchy-Schwartz

$$\begin{aligned}|\beta_n(\Delta_{Q+iP},\Delta_{\alpha_1},\Delta_{\alpha_2},\Delta_{\alpha_3},\Delta_{\alpha_4})|&\leq\beta_n(\Delta_{Q+iP},\Delta_{\alpha_1},\Delta_{\alpha_2},\Delta_{\alpha_2},\Delta_{\alpha_1})^{1/2}\beta_n(\Delta_{Q+iP},\Delta_{\alpha_4},\Delta_{\alpha_3},\Delta_{\alpha_3},\Delta_{\alpha_4})^{1/2}\\&\leq\frac{1}{2}(\beta_n(\Delta_{Q+iP},\Delta_{\alpha_1},\Delta_{\alpha_2},\Delta_{\alpha_2},\Delta_{\alpha_1})+\beta_n(\Delta_{Q+iP},\Delta_{\alpha_4},\Delta_{\alpha_3},\Delta_{\alpha_3},\Delta_{\alpha_4})).\end{aligned}$$

□

This lemma immediately implies for the conformal block the next result.

Lemma 4.5. For $|z| < 1$, we have:

$$|\mathcal{F}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(z,P)| \leq \frac{1}{2}\mathcal{F}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_2,\alpha_1}(|z|,P) + \frac{1}{2}\mathcal{F}^{\mathsf{sphere}}_{\alpha_4,\alpha_3,\alpha_3,\alpha_4}(|z|,P) \tag{4.8}$$

Now using this lemma we can bound:

$$\begin{aligned}
&\int_{\mathbb{R}} \left| \mathcal{M}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(P,P')(1-z)^{\frac{1}{2}(P')^2}\mathcal{F}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(1-z,P')\right| dP' \\
&\leq \frac{1}{2}\int_{\mathbb{R}} \left| \mathcal{M}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(P,P')(1-z)^{\frac{1}{2}(P')^2}\mathcal{F}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_1,\alpha_2}(|1-z|,P')\right| dP' \\
&+ \frac{1}{2}\int_{\mathbb{R}} \left| \mathcal{M}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(P,P')(1-z)^{\frac{1}{2}(P')^2}\mathcal{F}^{\mathsf{sphere}}_{\alpha_4,\alpha_3,\alpha_3,\alpha_4}(|1-z|,P')\right| dP'.
\end{aligned}$$

By using the result of Lemma A.3, one can see that these integrals are finite.

For the claim (b), we want to prove the same result but at $P = P_{m,n}$, after multiplying the block by the appropriate $(P - P_{m,n})^{m(\Delta_{r,s})}$ to cancel the pole. Establishing claim (b) thus follows exactly the same steps as for claim (a), except this we need to use the bound

$$\left| \frac{1}{\mathcal{Z}}(P-P_{m,n})^{m(\Delta_{r,s})}\mathcal{M}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(P,P')\right| \leq C_2 e^{c_2|P'|\log|P'|}$$

for some constants $c_2, C_2 > 0$ independent of everything and all P in a small neighborhood of $P_{m,n}$. We check using again Lemma A.3 that this bound holds.

Claim (c) of the theorem is a consequence of knowing from (cite [GKRV20]) that the power series expansion of $\mathcal{F}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(1-z,P')$ has the desired form. Lastly for claim (d) we will use the results on the residue of the fusion kernel established in the previous subsection. The starting point is the claim:

$$\mathrm{Res}_{P=P_{m,n}}\mathcal{M}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(P,P') = c(m,n,P)\mathcal{M}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(P_{-m,n},P'). \tag{4.9}$$

Taking the residue on both sides of the fusion transformation we obtain:

$$\begin{aligned}
&\mathrm{Res}_{P=P_{m,n}}\left(z^{\frac{1}{2}P^2}\mathcal{F}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(z,P)\right) \\
&= \mathrm{Res}_{P=P_{m,n}}\left(C(z)\int_{\mathbb{R}}\mathcal{M}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(P,P')(1-z)^{\frac{1}{2}(P')^2}\mathcal{F}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(1-z,P')dP'\right).
\end{aligned}$$

After exchanging the integral and the residue (justification should be the same as above), we get:

$$\begin{aligned}
&z^{\frac{1}{2}P_{m,n}^2}\mathrm{Res}_{P=P_{m,n}}\mathcal{F}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(z,P) \\
&= C(z)\int_{\mathbb{R}}\mathrm{Res}_{P=P_{m,n}}\mathcal{M}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(P,P')(1-z)^{\frac{1}{2}(P')^2}\mathcal{F}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(1-z,P')dP' \\
&= c(m,n,P)C(z)\int_{\mathbb{R}}\mathcal{M}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(P_{-m,n},P')(1-z)^{\frac{1}{2}(P')^2}\mathcal{F}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(1-z,P')dP' \\
&= c(m,n,P)z^{\frac{1}{2}P_{-m,n}^2}\mathcal{F}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(z,P_{-m,n}).
\end{aligned}$$

Plugging in the expression for $P_{m,n}$, this implies:

$$\mathrm{Res}_{P=P_{m,n}}\mathcal{F}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(z,P) = c(m,n,P)z^{2mn}\mathcal{F}^{\mathsf{sphere}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(z,P_{-m,n}).$$

□

5. The case of the 1-point torus conformal block

In this section we prove the analogous results to the ones established for the 4-point sphere conformal block in the case of the 1-pt torus conformal block. The outline of the section follows the same steps as for the sphere case. We start by deriving a GMC expression for the 1-point torus conformal block. Let us collect here some notations used in the two subsections below. First define:

$$\mathcal{A}_{\gamma,P,0}(\alpha) = e^{\frac{\mathbf{i}\pi\alpha^2}{2}}\left(\frac{\gamma}{2}\right)^{\frac{\gamma\alpha}{4}}e^{-\frac{\pi\alpha P}{2}}\Gamma(1-\frac{\gamma^2}{4})^{\frac{\alpha}{\gamma}}\frac{\Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2})\Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma}+\frac{\alpha}{2})\Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2}-\mathbf{i}P)\Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2}+\mathbf{i}P)}{\Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma})\Gamma_{\frac{\gamma}{2}}(Q-\mathbf{i}P)\Gamma_{\frac{\gamma}{2}}(Q+\mathbf{i}P)\Gamma_{\frac{\gamma}{2}}(Q-\alpha)}. \tag{5.1}$$

Under the parameter range $Q > \frac{\mathrm{Re}(\alpha)}{2}$, $\mathrm{Im}(P) \in \left[-Q + \frac{\mathrm{Re}(\alpha)}{2}, Q - \frac{\mathrm{Re}(\alpha)}{2}\right]$, define the contour integral

$$I_\alpha(P, P') := \int_{\mathcal{C}} d\xi \frac{S_{\frac{\gamma}{2}}(\frac{\mathbf{i}P'}{2} + \frac{\alpha}{4} + \xi) S_{\frac{\gamma}{2}}(\frac{\mathbf{i}P'}{2} + \frac{\alpha}{4} - \xi)}{S_{\frac{\gamma}{2}}(Q + \frac{\mathbf{i}P'}{2} - \frac{\alpha}{4} + \xi) S_{\frac{\gamma}{2}}(Q + \frac{\mathbf{i}P'}{2} - \frac{\alpha}{4} - \xi)} e^{2\pi P \xi},$$

where the contour $\mathcal{C}$ of the integral goes from $-\mathbf{i}\infty$ to $\mathbf{i}\infty$ passing to the right of the poles at $r = -\frac{\mathbf{i}P'}{2} - \frac{\alpha}{4} - n\frac{\gamma}{2} - m\frac{2}{\gamma}$, $r = \frac{\mathbf{i}P'}{2} - \frac{\alpha}{4} - n\frac{\gamma}{2} - m\frac{2}{\gamma}$ and to the left of the poles at $r = \frac{\mathbf{i}P'}{2} + \frac{\alpha}{4} + n\frac{\gamma}{2} + m\frac{2}{\gamma}$, $r = -\frac{\mathbf{i}P'}{2} + \frac{\alpha}{4} + n\frac{\gamma}{2} + m\frac{2}{\gamma}$, with $m, n \in \mathbb{N}^2$. The constraint on the parameters is again required in order for the contour integral to converge at $\pm\mathbf{i}\infty$. Then define the function

$$M_\alpha(P, P') := c_1 \frac{\sin(\frac{\gamma\pi\mathbf{i}P'}{2}) \sin(\frac{2\pi\mathbf{i}P'}{\gamma})}{S_{\frac{\gamma}{2}}(\frac{\alpha}{2})} I_\alpha(P, P'), \tag{5.2}$$

and finally the modular kernel $\mathcal{M}_\alpha^{\mathsf{torus}}(P, P')$ by the relation:

$$\mathcal{M}_\alpha^{\mathsf{torus}}(P, P') = M_\alpha(P, P') \frac{\mathcal{A}_{\gamma, P', 0}(\alpha)}{\mathcal{A}_{\gamma, P, 0}(\alpha)} e^{\frac{\pi\alpha(P' - P)}{2}}. \tag{5.3}$$

5.1. **GMC expressions of 1-point torus block.** As in the case of the 4-point sphere, we start by giving expressions for the 1-point torus block using GMC. Here there will actually be two different expressions, one coming from the boundary bootstrap as in the sphere case and one coming from our previous paper [GRSS20]. Let us start by giving the first of these definitions. Here let h_τ be the GFF on the annulus and define:

$$\hat{h}_\tau(x) = h_\tau(x) + \frac{\alpha}{2} \mathbb{E}[h_\tau(0) h_\tau(x)]. \tag{5.4}$$

Definition 5.1. (Probabilistic definitions of the 1-point torus block) Let $\alpha \in (0, Q)$ and $q \in (0, 1)$. For $P \in \mathbb{R}$ we can define:

$$\mathcal{G}_\alpha^{\mathsf{torus}}(q, P) := c_1 \mathbb{E}\left[\left(\int_0^1 e^{\frac{\gamma}{2}\hat{h}_\tau(x)} dx \right)^{-\frac{\alpha}{\gamma} - \frac{2iP}{\gamma}} \left(\int_{\frac{\tau}{2}}^{\frac{\tau}{2}+1} e^{\frac{\gamma}{2}\hat{h}_\tau(x)} dx \right)^{\frac{2iP}{\gamma}} \right].$$

$$\widetilde{\mathcal{G}}_\alpha^{\mathsf{torus}}(q, P) := c_2 \mathbb{E}\left[\left(\int_0^1 e^{\frac{\gamma}{2}\hat{h}_\tau(x)} e^{\gamma\pi P x} dx \right)^{-\frac{\alpha}{\gamma}} \right].$$

Notice that the main difference between the two expressions is in the location of the P parameter.

Theorem 5.2. Let $\alpha \in (0, Q)$, $P \in \mathbb{R}$ and $q \in (0, 1)$. Then one has:

$$\mathcal{F}_\alpha^{\mathsf{torus}}(q, P) = \mathcal{G}_\alpha^{\mathsf{torus}}(q, P) = \widetilde{\mathcal{G}}_\alpha^{\mathsf{torus}}(q, P) \quad \text{almost everywhere in } P \in \mathbb{R}. \tag{5.5}$$

To prove this theorem, we start by apply the operator $\mu_1^{-\frac{2\mathbf{i}P}{\gamma} - 1}$ to the right hand of the first bootstrap statement on the annulus.

Lemma 5.3. For $\alpha \in (0, Q)$, $q \in (0, 1)$, $P \in \mathbb{R}$, the following identity holds

$$q^{\frac{P^2}{2}} \mathcal{F}_\alpha^{\mathsf{torus}}(q, P) = \frac{1}{\mathcal{B}_1} \int_0^\infty d\mu_1 \mu_1^{-\frac{2\mathbf{i}P}{\gamma} - 1} \left(\int_{\mathbb{R}} G_{\mu_0}(\alpha, Q + \mathbf{i}P') U_{\mu_1}(Q - \mathbf{i}P') q^{\frac{1}{2}P'^2} \mathcal{G}_\alpha^{\mathsf{torus}}(q, P') dP' \right), \tag{5.6}$$

where the prefactor $\mathcal{B}_1$ is given by:

$$\mathcal{B}_1 = \Gamma(-\frac{2\mathbf{i}P}{\gamma}) \Gamma(1 - \frac{\mathbf{i}P\gamma}{2}) \mathcal{A}_{\gamma, P, 0}(\alpha) \mu_0^{-\frac{2\mathbf{i}P + \alpha}{\gamma}} \frac{2}{\gamma} e^{\frac{\pi\alpha P}{2} - \frac{\mathbf{i}\pi\alpha^2}{2}} (2\pi)^{-\frac{\alpha}{\gamma}} 2^{-2P^2 + \frac{\alpha}{2}(Q - \alpha) - \frac{\mathbf{i}P\alpha}{2}} (\mathbf{i}P) \Gamma(\frac{\alpha}{\gamma}) \Gamma(\frac{\mathbf{i}P\gamma}{2}) \Gamma(\frac{2\mathbf{i}P}{\gamma}) \frac{S_{\frac{\gamma}{2}}(\frac{\alpha}{2} + \mathbf{i}P)}{S_{\frac{\gamma}{2}}(\mathbf{i}P) S_{\frac{\gamma}{2}}(\frac{\alpha}{2})}.$$

Proof. The integral over P' on the right hand side of (5.6) using (2.5) can be viewed as $\int_{\mathbb{R}} \mu_1^{\frac{2\mathbf{i}P'}{\gamma}} g(P') dP'$ where:

$$g(P') := \frac{2}{\gamma} \Gamma(-\frac{2\mathbf{i}P'}{\gamma}) \left(\frac{2^{-\frac{\gamma}{2}(Q - \mathbf{i}P')} 2\pi}{\Gamma(1 - \frac{\gamma^2}{4})} \right)^{\frac{2\mathbf{i}P'}{\gamma}} \Gamma(1 - \frac{\mathbf{i}P'\gamma}{2}) G_{\mu_0}(\alpha, Q + \mathbf{i}P') q^{\frac{1}{2}P'^2} \mathcal{F}_\alpha^{\mathsf{torus}}(q, P').$$

Therefore applying the operator $\int_0^\infty d\mu_1 \mu_1^{-\frac{2\mathbf{i}P}{\gamma}-1}$ to the right hand side of (5.6) simply amounts to taking an inverse Fourier transform

$$\int_0^\infty d\mu_1 \mu_1^{-\frac{2\mathbf{i}P}{\gamma}-1} \int_{\mathbb{R}} dP' \mu_1^{\frac{2\mathbf{i}P'}{\gamma}} g(P') = \frac{\gamma}{2} \int_{\mathbb{R}} d\omega e^{-\mathbf{i}P\omega} \int_{\mathbb{R}} dP' e^{\mathbf{i}P'\omega} g(P') = \pi\gamma g(P).$$

It will be convenient to the express the result using $\mathcal{A}_{\gamma,P,0}(\alpha)$. For this purpose, record the expression

$$G_{\mu_0}(\alpha, Q+\mathbf{i}P) = \mu_0^{-\frac{2\mathbf{i}P+\alpha}{\gamma}} \frac{2}{\gamma}\Gamma(\frac{2\mathbf{i}P+\alpha}{\gamma}) \left(\frac{2\pi 2^{\frac{\gamma}{2}(\frac{\alpha}{2}-Q-\mathbf{i}P)}}{\Gamma(1-\frac{\gamma^2}{4})}\right)^{-\frac{2\mathbf{i}P+\alpha}{\gamma}} \frac{\Gamma(\frac{\alpha\gamma}{4}+\frac{\mathbf{i}\gamma P}{2}+1)\Gamma_{\frac{\gamma}{2}}(Q+\mathbf{i}P\pm\frac{\alpha}{2})\Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2})^2}{\Gamma_{\frac{\gamma}{2}}(Q-\alpha)\Gamma_{\frac{\gamma}{2}}(Q+\mathbf{i}P)^2\Gamma_{\frac{\gamma}{2}}(Q)},$$

where we adopt the convention that $\Gamma_{\frac{\gamma}{2}}(A\pm B) := \Gamma_{\frac{\gamma}{2}}(A+B)\Gamma_{\frac{\gamma}{2}}(A-B)$. By applying various identities, we can find that:

$$G_{\mu_0}(\alpha, Q+\mathbf{i}P) = \mathcal{A}_{\gamma,P,0}(\alpha)\mu_0^{-\frac{2\mathbf{i}P+\alpha}{\gamma}} \frac{2}{\gamma} e^{\frac{\pi\alpha P}{2}-\frac{\mathbf{i}\pi\alpha^2}{2}} (2\pi)^{-\frac{2\mathbf{i}P+\alpha}{\gamma}-1} 2^{-\frac{1}{2}(\alpha-Q-\mathbf{i}P)(2\mathbf{i}P+\alpha)} \Gamma(1-\frac{\gamma^2}{4})^{\frac{2\mathbf{i}P}{\gamma}}(\mathbf{i}P) \times \Gamma(\frac{\alpha}{\gamma})\Gamma(\frac{\mathbf{i}P\gamma}{2})\Gamma(\frac{2\mathbf{i}P}{\gamma}) \frac{S_{\frac{\gamma}{2}}(\frac{\alpha}{2}+\mathbf{i}P)}{S_{\frac{\gamma}{2}}(\mathbf{i}P)S_{\frac{\gamma}{2}}(\frac{\alpha}{2})}. \tag{5.7}$$

From here one obtains that

$$\pi\gamma g(P) = \Gamma(-\frac{2\mathbf{i}P}{\gamma})\Gamma(1-\frac{\mathbf{i}P\gamma}{2}) q^{\frac{1}{2}P^2} \mathcal{F}_\alpha^{\mathsf{torus}}(q,P)\mathcal{A}_{\gamma,P,0}(\alpha)\mu_0^{-\frac{2\mathbf{i}P+\alpha}{\gamma}} \frac{2}{\gamma} e^{\frac{\pi\alpha P}{2}-\frac{\mathbf{i}\pi\alpha^2}{2}} (2\pi)^{-\frac{\alpha}{\gamma}} 2^{-2P^2+\frac{\alpha}{2}(Q-\alpha)-\frac{\mathbf{i}P\alpha}{2}} (\mathbf{i}P) \times \Gamma(\frac{\alpha}{\gamma})\Gamma(\frac{\mathbf{i}P\gamma}{2})\Gamma(\frac{2\mathbf{i}P}{\gamma}) \frac{S_{\frac{\gamma}{2}}(\frac{\alpha}{2}+\mathbf{i}P)}{S_{\frac{\gamma}{2}}(\mathbf{i}P)S_{\frac{\gamma}{2}}(\frac{\alpha}{2})}$$

and thus the statement and the claimed expression for $\mathcal{B}_1$. □

We can now apply the same operator to the left hand side of the bootstrap statement to obtain the following lemma.

Lemma 5.4. Assume $\alpha \in (0,Q)$ and $P \in \mathbb{R}$. Then one has:

$$\int_0^\infty d\mu_1 \mu_1^{-\frac{2\mathbf{i}P}{\gamma}} \mathrm{LF}_\tau \left[L_1 e^{-\mu_0 L_0 - \mu_1 L_1} V_{\frac{\alpha}{2}}(0)\right] = \frac{2}{\gamma}\Gamma(-\frac{2\mathbf{i}P}{\gamma}+1)\Gamma(\frac{\alpha}{\gamma}+\frac{2\mathbf{i}P}{\gamma})\mathbb{E}\left[\mathcal{L}_{h_\tau}(\frac{\tau}{2},1+\frac{\tau}{2})^{\frac{2\mathbf{i}P}{\gamma}} \mathcal{L}_{h_\tau}(0,1)^{-\frac{2\mathbf{i}P}{\gamma}-\frac{\alpha}{\gamma}}\right]. \tag{5.8}$$

Proof. Let h_τ be given by equation (5.4). Denote by $\mathcal{L}_0 := \mathcal{L}_{h_\tau}(0,1)$ and $\mathcal{L}_1 := \mathcal{L}_{h_\tau}(\frac{\tau}{2}, 1+\frac{\tau}{2})$ the GMC measures of both boundaries of the annulus. By writing out explicitly the integral over the zero mode in the definition of the correlation function and by using the bounds for negative moments of GMC one gets:

$$\int_0^\infty d\mu_1 \int_{\mathbb{R}} dc \left|\mu_1^{-\frac{2\mathbf{i}P}{\gamma}} e^{\frac{(\alpha+\gamma)c}{2}} \mathbb{E}\left[\mathcal{L}_1 e^{-e^{\frac{\gamma c}{2}}\mu_0\mathcal{L}_0 - e^{\frac{\gamma c}{2}}\mu_1\mathcal{L}_1}\right]\right| = \frac{2}{\gamma}\Gamma(\frac{\alpha}{\gamma}+1)\int_0^\infty d\mu_1 \mathbb{E}\left[\mathcal{L}_1\left(\mu_0\mathcal{L}_0+\mu_1\mathcal{L}_1\right)^{-\frac{\alpha}{\gamma}-1}\right] < +\infty.$$

Thus we can compute:

$$\begin{aligned}
\int_0^\infty d\mu_1 \mu_1^{-\frac{2\mathbf{i}P}{\gamma}} \mathrm{LF}_\tau \left[L_1 e^{-\mu_0 L_0 - \mu_1 L_1} V_{\frac{\alpha}{2}}(0)\right] &= \int_0^\infty d\mu_1 \mu_1^{-\frac{2\mathbf{i}P}{\gamma}} \int_{\mathbb{R}} dc e^{\frac{(\alpha+\gamma)c}{2}} \mathbb{E}\left[\mathcal{L}_0 e^{-e^{\frac{\gamma c}{2}}\mu_0\mathcal{L}_0 - e^{\frac{\gamma c}{2}}\mu_1\mathcal{L}_1}\right] \\
&= \Gamma(-\frac{2\mathbf{i}P}{\gamma}+1)\int_{\mathbb{R}} dc e^{\frac{\alpha c}{2}} \mathbb{E}\left[(e^{\frac{\gamma c}{2}}\mathcal{L}_1)^{\frac{2\mathbf{i}P}{\gamma}} e^{-e^{\frac{\gamma c}{2}}\mu_0\mathcal{L}_0}\right] \\
&= \frac{2}{\gamma}\Gamma(-\frac{2\mathbf{i}P}{\gamma}+1)\Gamma(\frac{\alpha}{\gamma}+\frac{2\mathbf{i}P}{\gamma})\mathbb{E}[\mathcal{L}_1^{\frac{2\mathbf{i}P}{\gamma}} \mathcal{L}_0^{-\frac{2\mathbf{i}P}{\gamma}-\frac{\alpha}{\gamma}}].
\end{aligned}$$

□

5.2. **Proof of the modular equation.** Next we will prove the modular transformation in a smaller range of parameters than the one of our main result Theorem 1.4.

Theorem 5.5. For $\alpha \in (0,Q)$, $q \in (0,1)$, $P \in \mathbb{R}$:

$$q^{-\frac{1}{12}+\frac{1}{2}P^2}\mathcal{F}_\alpha^{\mathsf{torus}}(q,P) = \tau^{-\frac{\alpha}{2}(Q-\frac{\alpha}{2})}\int_{\mathbb{R}} \mathcal{M}_\alpha^{\mathsf{torus}}(P,P')\tilde{q}^{-\frac{1}{12}+\frac{1}{2}(P')^2}\mathcal{F}_\alpha^{\mathsf{torus}}(\tilde{q},P')dP'. \tag{5.9}$$

The proof is again based on applying again the operator $\mu_1^{-\frac{2\mathbf{i}P}{\gamma}-1}$ but to the second bootstrap statement on the annulus. We give this computation in the following proposition.

Proposition 5.6. For $\alpha \in (0, Q)$, $q \in (0,1)$, $P \in \mathbb{R}$, the following identity holds:
(5.10)

$$\int_{\mathbb{R}} \mathcal{M}^{\mathsf{torus}}_{\alpha}(P,P')\widetilde{q}^{\frac{1}{2}(P')^2}\mathcal{F}^{\mathsf{torus}}_{\alpha}(\widetilde{q},P')dP' = \frac{1}{\mathcal{B}_2}\int_0^\infty d\mu_1 \mu_1^{-\frac{2\mathbf{i}P}{\gamma}-1}\left(\int_{\mathbb{R}} H^{(\alpha,Q+\mathbf{i}P',Q-\mathbf{i}P')}_{(\mu_0,\mu_0,\mu_1)}\widetilde{q}^{\frac{1}{2}P'^2}\mathcal{F}^{\mathsf{torus}}_{\alpha}(\widetilde{q},P')dP'\right).$$

where the pre-factor $\mathcal{B}_2$ is given by:

$$\mathcal{B}_2 = 2\mathbf{i}\pi\mu_0^{-\frac{\alpha}{\gamma}-\frac{2\mathbf{i}P}{\gamma}}\frac{(2\pi)^{-\frac{\alpha}{\gamma}+1}(\frac{2}{\gamma})^{(\frac{\gamma}{2}-\frac{2}{\gamma})(-\frac{\alpha}{2})-1}}{\Gamma(1-\frac{\gamma^2}{4})^{-\frac{\alpha}{\gamma}}}\frac{\Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2})\Gamma_{\frac{\gamma}{2}}(\frac{\alpha}{2})\Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2}+\mathbf{i}P')}{\Gamma_{\frac{\gamma}{2}}(Q)\Gamma_{\frac{\gamma}{2}}(Q-\alpha)\Gamma_{\frac{\gamma}{2}}(\mathbf{i}P')\Gamma_{\frac{\gamma}{2}}(-\mathbf{i}P')}\frac{\Gamma_{\frac{\gamma}{2}}(-\mathbf{i}P'+Q-\frac{\alpha}{2})S_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2})}{S_{\frac{\gamma}{2}}(Q+\mathbf{i}P)S_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2}-\mathbf{i}P)}$$
$$\times\frac{1}{c_1}\frac{\mathcal{A}_{\gamma,P,0}(\alpha)}{\mathcal{A}_{\gamma,P',0}(\alpha)}e^{\frac{\pi\alpha(P-P')}{2}}\frac{1}{\sin(\frac{\gamma\pi\mathbf{i}P'}{2})\sin(\frac{2\pi\mathbf{i}P'}{\gamma})}.$$

We postpone the proof of this proposition and use it to finish the proof of the modular equation.

Proof. We thus equate the two bootstrap statements:

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}G_{\mu_0}(\alpha,Q+\mathbf{i}P)\partial_{\mu_1}U_{\mu_1}(Q-\mathbf{i}P)q^{\frac{1}{2}P^2}q^{-\frac{1}{12}}\mathcal{F}^{\mathsf{torus}}_{\alpha}(q,P)dP$$
$$=\frac{C_1}{2\pi}\tau^{-\frac{\alpha}{2}(Q-\frac{\alpha}{2})}\int_{-\infty}^{\infty}H^{(\alpha,Q+\mathbf{i}P,Q-\mathbf{i}P)}_{(\mu_0,\mu_0,\mu_1)}\widetilde{q}^{\frac{1}{2}P^2}\widetilde{q}^{-\frac{1}{12}}\mathcal{F}^{\mathsf{torus}}_{\alpha}(\widetilde{q},P)dP.$$

Applying various identities, we simplify the following expression:

$$\frac{\mathcal{B}_2}{\mathcal{B}_1} = 4\mathbf{i}\pi^2(\frac{2}{\gamma})^{(\frac{\gamma}{2}-\frac{2}{\gamma})(-\frac{\alpha}{2})-1}\frac{\Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2})\Gamma_{\frac{\gamma}{2}}(\frac{\alpha}{2})\Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2}+\mathbf{i}P')}{\Gamma_{\frac{\gamma}{2}}(Q)\Gamma_{\frac{\gamma}{2}}(Q-\alpha)\Gamma_{\frac{\gamma}{2}}(\mathbf{i}P')\Gamma_{\frac{\gamma}{2}}(-\mathbf{i}P')}\frac{\Gamma_{\frac{\gamma}{2}}(-\mathbf{i}P'+Q-\frac{\alpha}{2})}{S_{\frac{\gamma}{2}}(Q+\mathbf{i}P)S_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2}-\mathbf{i}P)}$$
$$\times\frac{1}{c_1}\frac{1}{\sin(\frac{\gamma\pi\mathbf{i}P'}{2})\sin(\frac{2\pi\mathbf{i}P'}{\gamma})}\left(\frac{\gamma}{2}\right)^{-\frac{\gamma\alpha}{4}}\frac{\Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma})\Gamma_{\frac{\gamma}{2}}(Q-\mathbf{i}P')\Gamma_{\frac{\gamma}{2}}(Q+\mathbf{i}P')\Gamma_{\frac{\gamma}{2}}(Q-\alpha)}{\Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2})\Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma}+\frac{\alpha}{2})\Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2}-\mathbf{i}P')\Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2}+\mathbf{i}P')}$$
$$\times\left(\Gamma(-\frac{2\mathbf{i}P}{\gamma})\Gamma(1-\frac{\mathbf{i}P\gamma}{2})\frac{2}{\gamma}2^{-2P^2+\frac{\alpha}{2}(Q-\alpha)-\frac{\mathbf{i}P\alpha}{2}}(\mathbf{i}P)\Gamma(\frac{\alpha}{\gamma})\Gamma(\frac{\mathbf{i}P\gamma}{2})\Gamma(\frac{2\mathbf{i}P}{\gamma})\frac{S_{\frac{\gamma}{2}}(\frac{\alpha}{2}+\mathbf{i}P)}{S_{\frac{\gamma}{2}}(\mathbf{i}P)}\right)^{-1}$$
$$=4\mathbf{i}(\frac{2}{\gamma})^{(\frac{\gamma}{2}-\frac{2}{\gamma})(-\frac{\alpha}{2})-1}\frac{1}{c_1}\left(\frac{\gamma}{2}\right)^{-\frac{\gamma\alpha}{4}}\left(\frac{2}{\gamma}2^{-2P^2+\frac{\alpha}{2}(Q-\alpha)-\frac{\mathbf{i}P\alpha}{2}}(\mathbf{i}P)\right)^{-1}\frac{P}{2\pi\mathbf{i}}\left(\frac{2}{\gamma}\right)^{-\frac{\alpha}{\gamma}+2}$$
$$=4\frac{1}{c_1}\left(2^{-2P^2+\frac{\alpha}{2}(Q-\alpha)-\frac{\mathbf{i}P\alpha}{2}}\mathbf{i}\right)^{-1}\frac{1}{2\pi}$$

This power of 2 can from the structure constants, maybe just remove it.. then one gets $c_1 = \frac{\mathbf{i}\pi}{2}$.

□

We now complete the proof of Proposition 5.6. We first prove the following integral identity expressing the integral transform of the H-function as a certain hyperbolic hypergeometric integral. In what follows, we will transform the resulting integral into the desired form using identities on hyperbolic hypergeometric integrals summarized in Appendix B. We divide the proof into two steps. In the first step, we express the integral transform of the H-function as a certain hypergeometric integral. In the second step, we identify the hypergeometric integral with the modular kernel up to some explicit pre-factor. Let us look at the analytic issues encountered in proving Proposition 3.3. The first thing we will do is apply the Fourier to the H function. At the end we will need to justify the inversion with the integration over P'. For this purpose we first state the following lemma.

Lemma 5.7. Consider $\alpha \in (0,Q)$, $P \in \mathbb{R}+\mathbf{i}\eta$ for a fixed $\eta > 0$. Then the following identity holds

$$\int_0^\infty \mu_1^{-\frac{2\mathbf{i}P}{\gamma}-1}H^{(\alpha,Q+\mathbf{i}P',Q-\mathbf{i}P')}_{(\mu_0,\mu_0,\mu_1)}d\mu_1 = \mathcal{B}_3\int_{\mathcal{C}_\delta}\frac{S_{\frac{\gamma}{2}}(\frac{\alpha}{2}+r)S_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2}+r)S_{\frac{\gamma}{2}}(\mathbf{i}P-r)}{S_{\frac{\gamma}{2}}(Q+\mathbf{i}P'+r)S_{\frac{\gamma}{2}}(Q+r)}e^{\frac{\mathbf{i}\pi}{2}(r-Q+2\mathbf{i}(P'-P))}dr,$$

where $\mathcal{B}_3$ is given by

$$\mathcal{B}_3 = 2\mathbf{i}\pi\mu_0^{-\frac{\alpha}{\gamma}-\frac{2\mathbf{i}P}{\gamma}}\frac{(2\pi)^{-\frac{\alpha}{\gamma}+1}(\frac{2}{\gamma})^{(\frac{\gamma}{2}-\frac{2}{\gamma})(-\frac{\alpha}{2})-1}}{\Gamma(1-\frac{\gamma^2}{4})^{-\frac{\alpha}{\gamma}}}\frac{\Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2})\Gamma_{\frac{\gamma}{2}}(\mathbf{i}P'+\frac{\alpha}{2})\Gamma_{\frac{\gamma}{2}}(\frac{\alpha}{2})\Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2}+\mathbf{i}P')}{\Gamma_{\frac{\gamma}{2}}(Q)\Gamma_{\frac{\gamma}{2}}(Q-\alpha)\Gamma_{\frac{\gamma}{2}}(\mathbf{i}P')\Gamma_{\frac{\gamma}{2}}(-\mathbf{i}P')}\frac{e^{\mathbf{i}\frac{\pi}{2}(\frac{\alpha}{2}(\frac{\alpha}{2}-Q))}e^{\mathbf{i}\pi PP'}}{S_{\frac{\gamma}{2}}(\frac{\alpha}{2})S_{\frac{\gamma}{2}}(Q+\mathbf{i}P)},$$

and where the contour $\mathcal{C}_\delta$ goes from $\pm \mathbf{i}\infty - \delta$ for a fix δ satisfying $\delta > \eta > 0$.

Proof. We start by scaling μ_0:

$$H^{(\alpha,Q+\mathbf{i}P',Q-\mathbf{i}P')}_{(\mu_0,\mu_0,\mu_1)} = \frac{2}{\gamma}\Gamma(\frac{\alpha}{\gamma})\overline{H}^{(\alpha,Q+\mathbf{i}P',Q-\mathbf{i}P')}_{(\mu_0,\mu_0,\mu_1)} = \frac{2}{\gamma}\Gamma(\frac{\alpha}{\gamma})\mu_0^{-\frac{\alpha}{\gamma}}\overline{H}^{(\alpha,Q+\mathbf{i}P',Q-\mathbf{i}P')}_{(1,1,\mu_1/\mu_0)}.$$

Therefore:

$$\begin{aligned}\int_0^\infty \mu_1^{-\frac{2\mathbf{i}P}{\gamma}-1} H^{(\alpha,Q+\mathbf{i}P',Q-\mathbf{i}P')}_{(\mu_0,\mu_0,\mu_1)} d\mu_1 &= \frac{2}{\gamma}\Gamma(\frac{\alpha}{\gamma})\mu_0^{-\frac{\alpha}{\gamma}}\int_0^\infty \mu_1^{-\frac{2\mathbf{i}P}{\gamma}-1}\overline{H}^{(\alpha,Q+iP',Q-iP')}_{(1,1,\mu_1/\mu_0)} d\mu_1\\ &= 2i\pi\Gamma(\frac{\alpha}{\gamma})\mu_0^{-\frac{\alpha}{\gamma}-\frac{2\mathbf{i}P}{\gamma}}\int_{\mathbf{i}\mathbb{R}} e^{2\pi(\sigma_3-\frac{Q}{2})P}\overline{H}^{(\alpha,Q+iP',Q-iP')}_{(1,1,e^{i\pi\gamma(\sigma_3-\frac{Q}{2})})} d\sigma_3\\ &= 2i\pi\Gamma(\frac{\alpha}{\gamma})\mu_0^{-\frac{\alpha}{\gamma}-\frac{2\mathbf{i}P}{\gamma}}\int_{\mathbf{i}\mathbb{R}} e^{2\pi(\sigma_2-\frac{Q}{2})P}\overline{H}^{(Q+iP',Q-iP',\alpha)}_{(1,e^{i\pi\gamma(\sigma_2-\frac{Q}{2})},1)} d\sigma_2.\end{aligned}$$

In the last step we have applied a cyclic permutation of parameters. We now use the exact formula for $\overline{H}$ with $\beta_3 = \alpha$, $\beta_1 = Q + iP'$, $\beta_2 = Q - iP'$, $\sigma_1 = \sigma_3 = \frac{Q}{2}$, $\frac{\mu_1}{\mu_0} = e^{i\pi\gamma(\sigma_2 - \frac{Q}{2})}$. One gets:

$$\begin{aligned}\overline{H}^{(Q+iP',Q-iP',\alpha)}_{(1,e^{i\pi\gamma(\sigma_2-\frac{Q}{2})},1)} &= \frac{(2\pi)^{-\frac{\alpha}{\gamma}+1}(\frac{2}{\gamma})^{(\frac{\gamma}{2}-\frac{2}{\gamma})(-\frac{\alpha}{2})-1}}{\Gamma(1-\frac{\gamma^2}{4})^{-\frac{\alpha}{\gamma}}\Gamma(\frac{\alpha}{\gamma})}\frac{e^{i\frac{\pi}{2}(\frac{\alpha}{2}(\frac{\alpha}{2}-Q))}\Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2})\Gamma_{\frac{\gamma}{2}}(iP'+\frac{\alpha}{2})\Gamma_{\frac{\gamma}{2}}(\frac{\alpha}{2})\Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2}+iP')}{\Gamma_{\frac{\gamma}{2}}(Q)\Gamma_{\frac{\gamma}{2}}(Q-\alpha)\Gamma_{\frac{\gamma}{2}}(iP')\Gamma_{\frac{\gamma}{2}}(-iP')S_{\frac{\gamma}{2}}(Q+\frac{iP'}{2}-\sigma_2)S_{\frac{\gamma}{2}}(\frac{\alpha}{2})}\\ &\times\int_{\mathcal{C}}\frac{S_{\frac{\gamma}{2}}(Q+\frac{iP'}{2}-\sigma_2+r)S_{\frac{\gamma}{2}}(\frac{\alpha}{2}+r)S_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2}+r)}{S_{\frac{\gamma}{2}}(Q+iP'+r)S_{\frac{\gamma}{2}}(Q+r)S_{\frac{\gamma}{2}}(Q+r)}e^{i\pi(\frac{iP'}{2}-Q+\sigma_2)r}\frac{dr}{i}.\end{aligned}$$

Let us check the parameter in order for the integration defining $\overline{H}$ to be well defined. The condition is $\mathrm{Re}(Q - \sigma_3 + \sigma_2 - \frac{\beta_2}{2}) > 0$ which translates given our values of β_i, σ_i to the condition $\mathrm{Re}(\sigma_2) > 0$. For this purpose we will consider the following quantity

$$\lim_{\epsilon\to 0}\int_{\mathbf{i}\mathbb{R}+\epsilon} e^{2\pi(\sigma_2-\frac{Q}{2})P}\overline{H}^{(Q+iP',Q-iP',\alpha)}_{(1,e^{i\pi\gamma(\sigma_2-\frac{Q}{2})},1)} d\sigma_2,$$

where at fixed $\epsilon > 0$ it is possible to use the above integral expression for $\overline{H}$. The next step will be to justify that we can exchange the order of integration of the two integrals over σ_2 and r.

For this purpose we will also consider a small modification of the contour $\mathcal{C}$ over the r variable. For a small $\delta > 0$, we consider $\mathcal{C}_\delta$ which goes this time from $\pm\mathbf{i}\infty - \delta$, and in between passes to the left and right of the poles of the integrand similarly as $\mathcal{C}$.

We will now show the following lemma that allows to exchange the order of integration:

For $\alpha \in (0, Q)$, $P' \in \mathbb{R}$, $P \in \mathbb{R} + \mathbf{i}\eta$ and for fixed η, ϵ, δ satisfying $\epsilon > 0$, $\delta > \eta > 0$ one has

$$\int_{\mathbf{i}\mathbb{R}+\epsilon} d\sigma_2 \int_{\mathcal{C}_\delta} dr\, |\mathcal{I}(r,\sigma_2)| < +\infty, \tag{5.11}$$

where we have used the notation:

$$\mathcal{I}(r,\sigma_2) := \frac{e^{2\pi(\sigma_2-\frac{Q}{2})P}}{S_{\frac{\gamma}{2}}(Q+\frac{iP'}{2}-\sigma_2)}\frac{S_{\frac{\gamma}{2}}(Q+\frac{iP'}{2}-\sigma_2+r)S_{\frac{\gamma}{2}}(\frac{\alpha}{2}+r)S_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2}+r)}{S_{\frac{\gamma}{2}}(Q+iP'+r)S_{\frac{\gamma}{2}}(Q+r)S_{\frac{\gamma}{2}}(Q+r)}e^{i\pi(\frac{iP'}{2}-Q+\sigma_2)r}.$$

For proving (5.11), it suffices to show that $|\mathcal{I}(r,\sigma_2)|$ decays exponentially as $\mathrm{Im}(r)$ or $\mathrm{Im}(\sigma_2)$ goes to $+\infty$ or $-\infty$. To this end, we use the asymptotics of the double sine function as noted in (B.1). Recall α, P' are real. We note that

$$|e^{i\pi(\frac{iP'}{2}-Q+\sigma_2)r}| = e^{\frac{\pi P'}{2}\delta}e^{\pi\mathrm{Im}(r)(Q-\epsilon)}e^{\pi\mathrm{Im}(\sigma_2)\delta}, \quad |S_{\frac{\gamma}{2}}(\sigma_2 - \frac{\mathbf{i}P'}{2})| \sim \begin{cases} e^{\pi(\epsilon-\frac{Q}{2})\mathrm{Im}(\sigma_2)} & \text{as } \mathrm{Im}(\sigma_2)\to\infty,\\ e^{-\pi(\epsilon-\frac{Q}{2})\mathrm{Im}(\sigma_2)} & \text{as } \mathrm{Im}(\sigma_2)\to-\infty,\end{cases}$$

and,

$$|S_{\frac{\gamma}{2}}(Q+\frac{iP'}{2}-\sigma_2+r)| \sim \begin{cases} c_1 e^{\pi(\frac{Q}{2}-\delta-\epsilon)\mathrm{Im}(r-\sigma_2)} & \text{as } \mathrm{Im}(r-\sigma_2)\to\infty,\\ c_2 e^{-\pi(\frac{Q}{2}-\delta-\epsilon)\mathrm{Im}(r-\sigma_2)} & \text{as } \mathrm{Im}(r-\sigma_2)\to-\infty.\end{cases}$$

$$\left|\frac{S_{\frac{\gamma}{2}}(\frac{\alpha}{2}+r)S_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2}+r)}{S_{\frac{\gamma}{2}}(Q+iP'+r)S_{\frac{\gamma}{2}}(Q+r)S_{\frac{\gamma}{2}}(Q+r)}\right| \sim \begin{cases} e^{\pi(\delta-\frac{3}{2}Q)\mathrm{Im}(r)} & \text{as } \mathrm{Im}(r)\to\infty,\\ e^{-\pi(\delta-\frac{3}{2}Q)\mathrm{Im}(r)} & \text{as } \mathrm{Im}(r)\to-\infty.\end{cases} \tag{5.12}$$

By our assumption, $P = \mathrm{Re}(P) + \mathbf{i}\eta$ which implies $|\exp(2\pi(\sigma_2 - \frac{Q}{2})P)| \sim \exp(-2\pi \mathrm{Im}(\sigma_2)\eta)$. In what follows, we use the above asymptotics to find the decay of $\mathcal{I}(r,\sigma_2)$ as $\mathrm{Im}(r), \mathrm{Im}(\sigma_2) \to \pm\infty$.

In what follows, we study the asymptotics of $\mathcal{I}(r,\sigma_2)$ in eight different possible cases based on the relative growth of $\mathrm{Im}(r)$ and $\mathrm{Im}(\sigma_2)$. In all cases, we use the asymptotics such that there

- $\mathrm{Im}(r) \to +\infty$, $\mathrm{Im}(\sigma_2) \to +\infty$, $\mathrm{Im}(r-\sigma_2) \to +\infty$
$$|\mathcal{I}(r,\sigma_2)| \sim e^{-2\pi \mathrm{Im}(r)\epsilon} e^{\pi \mathrm{Im}(\sigma_2)(2\delta+2\epsilon-2\eta-Q)},$$
- $\mathrm{Im}(r) \to +\infty$, $\mathrm{Im}(\sigma_2) \to +\infty$, $\mathrm{Im}(r-\sigma_2) \to -\infty$
$$|\mathcal{I}(r,\sigma_2)| \sim e^{\pi \mathrm{Im}(r)(2\delta-Q)} e^{-2\pi \mathrm{Im}(\sigma_2)\eta},$$
- $\mathrm{Im}(r) \to +\infty$, $\mathrm{Im}(\sigma_2) \to +\infty$, $\mathrm{Im}(r-\sigma_2) \to C$
$$|\mathcal{I}(r,\sigma_2)| \sim e^{\pi \mathrm{Im}(r)(\delta-\epsilon-\frac{Q}{2})} e^{\pi \mathrm{Im}(\sigma_2)(\delta+\epsilon-\frac{Q}{2}-2\eta)}$$
- $\mathrm{Im}(r) \to -\infty$, $\mathrm{Im}(\sigma_2) \to -\infty$, $\mathrm{Im}(r-\sigma_2) \to +\infty$
$$|\mathcal{I}(r,\sigma_2)| \sim e^{\pi \mathrm{Im}(r)(3Q-2\epsilon-2\delta)} e^{\pi \mathrm{Im}(\sigma_2)(2\delta-2\eta)}$$
- $\mathrm{Im}(r) \to -\infty$, $\mathrm{Im}(\sigma_2) \to -\infty$, $\mathrm{Im}(r-\sigma_2) \to -\infty$
$$|\mathcal{I}(r,\sigma_2)| \sim e^{2\pi \mathrm{Im}(r)Q} e^{\pi \mathrm{Im}(\sigma_2)(Q-2\epsilon-2\eta)}$$
- $\mathrm{Im}(r) \to -\infty$, $\mathrm{Im}(\sigma_2) \to -\infty$, $\mathrm{Im}(r-\sigma_2) \to C$
$$|\mathcal{I}(r,\sigma_2)| \sim e^{\pi \mathrm{Im}(r)(\frac{5}{2}Q-\epsilon-\delta)} e^{\pi \mathrm{Im}(\sigma_2)(\delta-\epsilon+\frac{Q}{2}-2\eta)}$$
- $\mathrm{Im}(r) \to +\infty$, $\mathrm{Im}(\sigma_2) \to -\infty$, $\mathrm{Im}(r-\sigma_2) \to +\infty$
$$|\mathcal{I}(r,\sigma_2)| \sim e^{-2\pi \mathrm{Im}(r)\epsilon} e^{\pi \mathrm{Im}(\sigma_2)(2\delta-2\eta)}$$
- $\mathrm{Im}(r) \to -\infty$, $\mathrm{Im}(\sigma_2) \to +\infty$, $\mathrm{Im}(r-\sigma_2) \to -\infty$
$$|\mathcal{I}(r,\sigma_2)| \sim e^{2\pi \mathrm{Im}(r)Q} e^{-2\pi \mathrm{Im}(\sigma_2)\eta}$$

Combining the asymptotics in all of these above cases shows that for $\epsilon > 0$ and $\delta > \eta > 0$, there exists $C, c_1, c_2 > 0$ such that
$$|\mathcal{I}(r,\sigma_2)| \leq C e^{-c_1|\mathrm{Im}(r)| - c_2|\mathrm{Im}(\sigma_2)|}.$$
Now (5.11) follows from the above display via Tonelli's theorem.

Using the above lemma we can exchange the integration over r and σ_2 and compute the integral over σ_2:
$$\begin{aligned}
&\int_{\mathbf{i}\mathbb{R}+\epsilon} d\sigma_2 e^{2\pi(\sigma_2-\frac{Q}{2})P} e^{i\pi(\frac{iP'}{2}-Q+\sigma_2)r} \frac{S_{\frac{\gamma}{2}}(Q+\frac{iP'}{2}-\sigma_2+r)}{S_{\frac{\gamma}{2}}(Q+\frac{iP'}{2}-\sigma_2)} \\
&= e^{-\pi QP-\pi P'r+i\pi PP'} e^{-i\pi Qr} \int_{\mathbf{i}\mathbb{R}+\epsilon} d\sigma_2 e^{i\pi\sigma_2(2iP-r)} \frac{S_{\frac{\gamma}{2}}(Q+\sigma_2+r)}{S_{\frac{\gamma}{2}}(Q+\sigma_2)} \\
&= ie^{-\pi QP-\pi P'r+i\pi PP'} e^{-i\pi Qr} e^{\frac{i\pi}{2}(\alpha Q-\alpha^2)} e^{-\pi i\alpha\beta} \frac{S_{\frac{\gamma}{2}}(Q+r)S_{\frac{\gamma}{2}}(iP-r)}{S_{\frac{\gamma}{2}}(Q+iP)} \\
&= ie^{-\pi P'r+i\pi PP'} e^{\frac{i\pi}{2}(-Qr+r^2-2riP)} \frac{S_{\frac{\gamma}{2}}(Q+r)S_{\frac{\gamma}{2}}(iP-r)}{S_{\frac{\gamma}{2}}(Q+iP)},
\end{aligned}$$
where in the second equality we shift the integration over σ_2 (which is clearly valid since P' is real) and in the third equality we use the hyperbolic beta integral from Lemma B.1 with $\alpha = Q + r$, $\beta = \mathbf{i}P - r$. The three conditions $\mathrm{Re}(Q + r) > 0$, $\mathrm{Re}(\mathbf{i}P - r) > 0$, $\mathrm{Re}(Q + \mathbf{i}P) < Q$ are valid since $r \in \mathcal{C}_\delta$, $P \in \mathbb{R} + \mathbf{i}\eta$ and $\delta > \eta > 0$. Putting everything together we obtain the desired claim.

Finally lets us check here the convergence at $r \to \pm\mathbf{i}\infty$ of the integral in the right hand side of Lemma 5.7. We compute the asymptotic of the double sine function in the integrand:

$$\frac{S_{\frac{\gamma}{2}}(\frac{\alpha}{2}+r)S_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2}+r)}{S_{\frac{\gamma}{2}}(Q+iP'+r)S_{\frac{\gamma}{2}}(Q+r)S_{\frac{\gamma}{2}}(Q-iP+r)} \sim \begin{cases} c_1 e^{-\mathbf{i}\frac{\pi}{2}\Delta_1} & \text{as} \quad \mathrm{Im}(r) \to \infty, \\ c_2 e^{\mathbf{i}\frac{\pi}{2}\Delta_1} & \text{as} \quad \mathrm{Im}(r) \to -\infty. \end{cases} \tag{5.13}$$

where here $\Delta_1 = -r^2 + r(2\mathbf{i}P + 2\mathbf{i}P' - 3Q)$. Therefore:

(5.14)
$$\frac{S_{\frac{\gamma}{2}}(\frac{\alpha}{2}+r)S_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2}+r)}{S_{\frac{\gamma}{2}}(Q+iP'+r)S_{\frac{\gamma}{2}}(Q+r)S_{\frac{\gamma}{2}}(Q-iP+r)}e^{-\pi P' r}e^{\frac{i\pi}{2}(-Qr+r^2-2riP)} \sim \begin{cases} c_1 e^{-\mathbf{i}\frac{\pi}{2}\Delta_2} & \text{as} \quad \mathrm{Im}(r) \to \infty, \\ c_2 e^{\mathbf{i}\frac{\pi}{2}\Delta_3} & \text{as} \quad \mathrm{Im}(r) \to -\infty. \end{cases}$$

Here $\Delta_2 = -2r^2 + r(4\mathbf{i}P - 2Q)$ and $\Delta_3 = r(4\mathbf{i}P' - 4Q)$. We assume that P' is real but that P can have a small imaginary part $\mathbf{i}\eta$. Also let us still assume that $r = \mathbf{i}\mathrm{Im}(r) - \delta$. Based on the values of Δ_2, Δ_3, the integral is converging. □

We will now express the integral in the right hand side of Lemma 5.7 as a constant times the modular kernel. This is given by the following lemma:

Lemma 5.8. Consider $\alpha \in (0, Q)$, $P \in \mathbb{R} + \mathbf{i}\eta$ for a fixed $\eta > 0$. Then the following identity holds
$$\mathcal{M}_\alpha^{\mathsf{torus}}(P, P') = \frac{1}{\mathcal{B}_2}\int_0^\infty \mu_1^{-\frac{2\mathbf{i}P}{\gamma}-1} H^{(\alpha, Q+\mathbf{i}P', Q-\mathbf{i}P')}_{(\mu_0,\mu_0,\mu_1)} d\mu_1,$$
where $\mathcal{B}_2$ is as in Proposition 5.6.

Proof. Applying Lemma B.7 to the output of Lemma 5.7, we find that:
$$\int_0^\infty \mu_1^{-\frac{2\mathbf{i}P}{\gamma}-1} H^{(\alpha, Q+\mathbf{i}P', Q-\mathbf{i}P')}_{(\mu_0,\mu_0,\mu_1)} d\mu_1 = \mathcal{B}_3 e^{\frac{\mathbf{i}\pi}{2}(\Delta_\alpha - 2PP')} \frac{S_{\frac{\gamma}{2}}(-\mathbf{i}P' + Q - \frac{\alpha}{2})S_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2})}{S_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2}-\mathbf{i}P)} I_\alpha(P, P').$$
On the other hand we also have that:
$$\mathcal{M}_\alpha^{\mathsf{torus}}(P, P') = c_1 \frac{\mathcal{A}_{\gamma, P', 0}(\alpha)}{\mathcal{A}_{\gamma, P, 0}(\alpha)} e^{\frac{\pi\alpha(P'-P)}{2}} \frac{\sin(\frac{\gamma\pi\mathbf{i}P'}{2})\sin(\frac{2\pi\mathbf{i}P'}{\gamma})}{S_{\frac{\gamma}{2}}(\frac{\alpha}{2})} I_\alpha(P, P').$$
This implies the claim of the lemma. □

To finish the proof of Proposition 5.6 we need to justify the exchange of the integral over μ_1 and the integration over P'. This is provided to the following lemma.

Lemma 5.9. In the parameter range $\alpha \in (0, Q)$, $q \in (0, 1)$, $P \in \mathbb{R} + \mathbf{i}\eta$, $\eta > 0$ one has:
$$\int_0^\infty d\mu_1 \mu_1^{-\frac{2\mathbf{i}P}{\gamma}} \left(\int_{-\infty}^\infty H^{(\alpha, Q+\mathbf{i}P', Q-\mathbf{i}P')}_{(\mu_0,\mu_0,\mu_1)} \tilde{q}^{\frac{1}{2}P'^2} \mathcal{F}_\alpha^{\mathsf{torus}}(\tilde{q}, P') dP'\right)$$
$$= \int_{-\infty}^\infty dP' \left(\int_0^\infty d\mu_1 \mu_1^{-\frac{2\mathbf{i}P}{\gamma}} H^{(\alpha, Q+\mathbf{i}P', Q-\mathbf{i}P')}_{(\mu_0,\mu_0,\mu_1)} \tilde{q}^{\frac{1}{2}P'^2} \mathcal{F}_\alpha^{\mathsf{torus}}(\tilde{q}, P')\right).$$

Proof. One needs to show:
$$\int_{\mathbb{R}} dP' \int_0^\infty d\mu_1 \left| \mu_1^{-\frac{2\mathbf{i}P}{\gamma}} H^{(\alpha, Q+\mathbf{i}P', Q-\mathbf{i}P')}_{(\mu_0,\mu_0,\mu_1)} \tilde{q}^{\frac{1}{2}P'^2} \mathcal{F}_\alpha^{\mathsf{torus}}(\tilde{q}, P') \right| < +\infty.$$
To show this one simply needs to use the fact that
$$\lim_{\mu_1 \to 0} H^{(\alpha, Q+\mathbf{i}P', Q-\mathbf{i}P')}_{(\mu_0,\mu_0,\mu_1)} = H^{(\alpha, Q+\mathbf{i}P', Q-\mathbf{i}P')}_{(\mu_0,\mu_0,0)},$$
and the fact that:
$$H^{(\alpha, Q+\mathbf{i}P', Q-\mathbf{i}P')}_{(\mu_0,\mu_0,\mu_1)} = \mu_1^{-\frac{\alpha}{\gamma}} H^{(\alpha, Q+\mathbf{i}P', Q-\mathbf{i}P')}_{(\frac{\mu_0}{\mu_1},\frac{\mu_0}{\mu_1},1)} \underset{\mu_1 \to +\infty}{\sim} \mu_1^{-\frac{\alpha}{\gamma}} H^{(\alpha, Q+\mathbf{i}P', Q-\mathbf{i}P')}_{(0,0,1)}.$$
Using this asymptotic the integral over μ_1 is always convergent at 0 and at infinity it behaves as $C\mu_1^{-\frac{\alpha}{\gamma}+\frac{2\eta}{\gamma}}$ and so one needs to assume $\alpha \in (\gamma + 2\eta, Q)$ for it converge. For η this interval is non empty since one has $\gamma < Q$. One can then check the integration over P' is converging by using the bounds on the special functions. □

Finally we had assumed $P \in \mathbb{R} + \mathbf{i}\eta$, $\eta > 0$. Since the left hand side of (5.10) is continuous in P in a neighborhood of $\mathbb{R}$, one can clearly take the limit $\eta \to 0$ and obtain the desired result for $P \in \mathbb{R}$.

5.3. **Analytic properties of the modular kernel.** The modular kernel $\mathcal{M}_\alpha^{\mathsf{torus}}(P,P')$ given in (5.3) in the previous section involves the integral formula $I_\alpha(P,P')$ which is only converging in the parameter range $Q > \frac{\mathrm{Re}(\alpha)}{2}$, $-Q+\frac{\mathrm{Re}(\alpha)}{2} < \mathrm{Im}(P) < Q - \frac{\mathrm{Re}(\alpha)}{2}$. In order to extend the 1-point torus conformal block to a meromorphic function of P, it will first be necessary to extend the modular kernel to a meromorphic function of $P \in \mathbb{C}$, fixing all the other parameters in some appropriate range. We will also show the modular kernel has poles at $P = P_{m,n}$ with residues satisfying a certain identity. This is the content of the following proposition.

Proposition 5.10. Fix $\gamma \in (0,2)$, $\alpha \in (0,Q)$, and $P' \in \mathbb{R}$. The modular kernel $P \mapsto \mathcal{M}_\alpha(P,P')$ is a meromorphic function over $\mathbb{C}$ with poles at $P = P_{m,n}$. Furthermore it obeys:

$$\mathrm{Res}_{P=P_{m,n}} \mathcal{M}_\alpha^{\mathsf{torus}}(P,P') = d(m,n,P)\mathcal{M}_\alpha^{\mathsf{torus}}(P_{-m,n},P'). \tag{5.15}$$

In order to prove this result, we will show that the modular kernel obeys certain functional equations known as shift equations. Consider the function $M_\alpha(P,P')$ related to the fusion kernel by the relation:

$$\mathcal{M}_\alpha(P,P') = M_\alpha(P,P')\frac{\mathcal{A}_{\gamma,P',0}(\alpha)}{\mathcal{A}_{\gamma,P,0}(\alpha)}e^{\frac{\pi\alpha(P'-P)}{2}}.$$

We know give the following lemma on the function $M_\alpha(P,P')$.

Lemma 5.11. Fix $\alpha \in (0,Q)$ and $P' \in \mathbb{R}$. The function $P \to M_\alpha(P,P')$ can be meromorphically extended to $\mathbb{C}$. For both $\chi = \frac{\gamma}{2}$ or $\frac{2}{\gamma}$, it obeys the following functional equation:

$$\frac{\sin\pi\chi(\mathbf{i}P+\frac{\alpha}{2})}{\sin\pi\chi\mathbf{i}P}M_\alpha(P-\mathbf{i}\chi,P') + \frac{\sin\pi\chi(\mathbf{i}P-\frac{\alpha}{2})}{\sin\pi\chi\mathbf{i}P}M_\alpha(P+\mathbf{i}\chi,P') = 2\cos(\pi\chi\mathbf{i}P')M_\alpha(P,P').$$

Proof. Assume a function $f(P)$ is a solution to:

$$\sin(\pi\chi(\mathbf{i}P+\frac{\alpha}{2}))f(P-\mathbf{i}\chi) + \sin(\pi\chi(\mathbf{i}P-\frac{\alpha}{2}))f(P+\mathbf{i}\chi) = 2\cos(\pi\chi\mathbf{i}P')\sin(\pi\chi\mathbf{i}P)f(P).$$

Using Fourier transform we can write $f(P)$ as

$$f(P) = \int_{\mathcal{C}} d\xi e^{2\pi P\xi}\hat{f}(\xi)$$

where $\mathcal{C}$ is some appropriate contour going from $-\mathbf{i}\infty$ to $+\mathbf{i}\infty$ and now try to identify the function $\hat{f}(\xi)$. The above functional equation then becomes:

$$\begin{aligned}&\left(e^{\frac{\pi i\chi\alpha}{2}-\pi\chi P} - e^{-\frac{\pi i\chi\alpha}{2}+\pi\chi P}\right)\int_{\mathcal{C}} d\xi e^{2\pi(P-\mathbf{i}\chi)\xi}\hat{f}(\xi) + \left(e^{-\frac{\pi i\chi\alpha}{2}-\pi\chi P} - e^{\frac{\pi i\chi\alpha}{2}+\pi\chi P}\right)\int_{\mathcal{C}} d\xi e^{2\pi(P+\mathbf{i}\chi)\xi}\hat{f}(\xi)\\ &= 2\cos(\pi\chi\mathbf{i}P')\left(e^{-\pi\chi P}-e^{\pi\chi P}\right)\int_{\mathcal{C}} d\xi e^{2\pi P\xi}\hat{f}(\xi).\end{aligned}$$

We now move the contour $\mathcal{C}$ to get:

$$\begin{aligned}&\int_{\mathcal{C}} d\xi e^{2\pi P\xi}\hat{f}(\xi+\frac{\chi}{2})\left[e^{\frac{\pi\mathbf{i}\chi\alpha}{2}-2\pi\mathbf{i}\chi(\xi+\frac{\chi}{2})} + e^{-\frac{\pi\mathbf{i}\chi\alpha}{2}+2\pi\mathbf{i}\chi(\xi+\frac{\chi}{2})} - 2\cos(\pi\chi\mathbf{i}P')\right]\\ &-\int_{\mathcal{C}} d\xi e^{2\pi P\xi}\hat{f}(\xi-\frac{\chi}{2})\left[e^{\frac{\pi\mathbf{i}\chi\alpha}{2}+2\pi\mathbf{i}\chi(\xi-\frac{\chi}{2})} + e^{-\frac{\pi\mathbf{i}\chi\alpha}{2}-2\pi\mathbf{i}\chi(\xi-\frac{\chi}{2})} - 2\cos(\pi\chi\mathbf{i}P')\right] = 0.\end{aligned}$$

By taking the inverse Fourier transform this implies the following equation on $\hat{f}(\xi)$:

$$\begin{aligned}\hat{f}(\xi+\frac{\chi}{2}) &= \frac{\cos(\frac{\pi\chi\alpha}{2}+2\pi\chi(\xi-\frac{\chi}{2})) - \cos(\pi\chi\mathbf{i}P')}{\cos(\frac{\pi\chi\alpha}{2}-2\pi\chi(\xi+\frac{\chi}{2})) - \cos(\pi\chi\mathbf{i}P')}\hat{f}(\xi-\frac{\chi}{2})\\ &= \frac{\sin(\frac{\pi\chi\alpha}{4}+\pi\chi(\xi-\frac{\chi}{2})-\frac{1}{2}\pi\chi\mathbf{i}P')\sin(\frac{\pi\chi\alpha}{4}+\pi\chi(\xi-\frac{\chi}{2})+\frac{1}{2}\pi\chi\mathbf{i}P')}{\sin(\frac{\pi\chi\alpha}{4}-\pi\chi(\xi+\frac{\chi}{2})-\frac{1}{2}\pi\chi\mathbf{i}P')\sin(\frac{\pi\chi\alpha}{4}-\pi\chi(\xi+\frac{\chi}{2})+\frac{1}{2}\pi\chi\mathbf{i}P')}\hat{f}(\xi-\frac{\chi}{2})\\ \Rightarrow \hat{f}(\xi+\chi) &= \frac{\sin(\frac{\pi\chi\alpha}{4}+\pi\chi\xi-\frac{1}{2}\pi\chi\mathbf{i}P')\sin(\frac{\pi\chi\alpha}{4}+\pi\chi\xi+\frac{1}{2}\pi\chi\mathbf{i}P')}{\sin(\frac{\pi\chi\alpha}{4}-\pi\chi(\xi+\chi)-\frac{1}{2}\pi\chi\mathbf{i}P')\sin(\frac{\pi\chi\alpha}{4}-\pi\chi(\xi+\chi)+\frac{1}{2}\pi\chi\mathbf{i}P')}\hat{f}(\xi).\end{aligned}$$

Lets check that the integrand of $I_\alpha(P,P')$ obeys the same relation. For this we will use the functional equation (A.6) on $S_{\frac{\gamma}{2}}$. Thus let:

$$g(\xi) = \frac{S_{\frac{\gamma}{2}}(\frac{\mathbf{i}P'}{2}+\frac{\alpha}{4}+\xi)S_{\frac{\gamma}{2}}(\frac{\mathbf{i}P'}{2}+\frac{\alpha}{4}-\xi)}{S_{\frac{\gamma}{2}}(Q+\frac{\mathbf{i}P'}{2}-\frac{\alpha}{4}+\xi)S_{\frac{\gamma}{2}}(Q+\frac{\mathbf{i}P'}{2}-\frac{\alpha}{4}-\xi)}.$$

Then we can compute that:

$$\begin{aligned} g(\xi+\chi) &= \frac{S_{\frac{\gamma}{2}}(\frac{\mathbf{i}P'}{2}+\frac{\alpha}{4}+\chi+\xi)S_{\frac{\gamma}{2}}(\frac{\mathbf{i}P'}{2}+\frac{\alpha}{4}-\chi-\xi)}{S_{\frac{\gamma}{2}}(Q+\frac{\mathbf{i}P'}{2}-\frac{\alpha}{4}+\chi+\xi)S_{\frac{\gamma}{2}}(Q+\frac{\mathbf{i}P'}{2}-\frac{\alpha}{4}-\chi-\xi)} \\ &= \frac{\sin(\frac{\pi\chi\mathbf{i}P'}{2}+\frac{\pi\chi\alpha}{4}+\pi\chi\xi)}{\sin(\pi\chi^2+\frac{\pi\chi\mathbf{i}P'}{2}-\frac{\pi\chi\alpha}{4}+\pi\chi\xi)}\frac{\sin(\frac{\pi\chi\mathbf{i}P'}{2}-\frac{\pi\chi\alpha}{4}-\pi\chi\xi)}{\sin(\frac{\pi\chi\mathbf{i}P'}{2}+\frac{\pi\chi\alpha}{4}-\pi\chi^2-\pi\chi\xi)}g(\xi) \end{aligned}$$

This matches the relation for $\hat{f}$ and thus we have checked the first shift equations. □

Remark. It is also possible to derive the following extra three shift equations:

$$\begin{aligned} &\frac{\sin\pi\chi(\mathbf{i}P'-\chi+\frac{\alpha}{2})}{\sin\pi\chi(\mathbf{i}P'-\chi)}M_\alpha(P,P'+\mathbf{i}\chi)+\frac{\sin\pi\chi(\mathbf{i}P'+\chi-\frac{\alpha}{2})}{\sin\pi\chi(\mathbf{i}P'+\chi)}M_\alpha(P,P'-\mathbf{i}\chi) = 2\cos(\pi\chi\mathbf{i}P)M_\alpha(P,P'), \\ &\frac{1}{\sin\pi\chi(\mathbf{i}P'-\chi)}M_{\alpha+2\chi}(P,P'+\mathbf{i}\chi)-\frac{1}{\sin\pi\chi(\mathbf{i}P'+\chi)}M_{\alpha+2\chi}(P,P'-\mathbf{i}\chi) = 2M_\alpha(P,P'), \\ &\frac{1}{\sin\pi\chi\mathbf{i}P}\left(M_\alpha(P-\mathbf{i}\chi,P')-M_\alpha(P+\mathbf{i}\chi,P')\right) = 2M_{\alpha+2\chi}(P,P'). \end{aligned}$$

As they will not be needed in the present paper, we omit their proofs.

Using the shift equation we have established, we can now prove Proposition 5.10.

Proof of Proposition 5.10. Fix $\alpha < Q$, $P' \in \mathbb{R}$. The goal is to show that the function given by

$$\begin{aligned} \mathcal{M}_\alpha(P,P') &= M_\alpha(P,P')\frac{\mathcal{A}_{\gamma,P',0}(\alpha)}{\mathcal{A}_{\gamma,P,0}(\alpha)}e^{\frac{\pi\alpha(P'-P)}{2}}, \\ M_\alpha(P,P') &:= c_1\frac{\sin(\frac{\gamma\pi\mathbf{i}P'}{2})\sin(\frac{2\pi\mathbf{i}P'}{\gamma})}{S_{\frac{\gamma}{2}}(\frac{\alpha}{2})}I_\alpha(P,P'), \\ I_\alpha(P,P') &:= \int_{\mathcal{C}} d\xi \frac{S_{\frac{\gamma}{2}}(\frac{\mathbf{i}P'}{2}+\frac{\alpha}{4}+\xi)S_{\frac{\gamma}{2}}(\frac{\mathbf{i}P'}{2}+\frac{\alpha}{4}-\xi)}{S_{\frac{\gamma}{2}}(Q+\frac{\mathbf{i}P'}{2}-\frac{\alpha}{4}+\xi)S_{\frac{\gamma}{2}}(Q+\frac{\mathbf{i}P'}{2}-\frac{\alpha}{4}-\xi)}e^{2\pi P\xi}, \end{aligned}$$

which is originally defined under the parameter constraint $\mathrm{Im}(P) \in [-Q+\frac{\mathrm{Re}(\alpha)}{2}, Q-\frac{\mathrm{Re}(\alpha)}{2}]$ extends to a meromorphic function in $P \in \mathbb{C}$ with poles at $P = P_{m,n}$ and residues prescribed by the stated relation. To prove this fact the two inputs will be the shift equation prove on $M_\alpha(P,P')$, namely

$$\frac{\sin\pi\chi(\mathbf{i}P+\frac{\alpha}{2})}{\sin\pi\chi\mathbf{i}P}M_\alpha(P-\mathbf{i}\chi,P')+\frac{\sin\pi\chi(\mathbf{i}P-\frac{\alpha}{2})}{\sin\pi\chi\mathbf{i}P}M_\alpha(P+\mathbf{i}\chi,P') = 2\cos(\pi\chi\mathbf{i}P')M_\alpha(P,P'),$$

and the fact that the P dependence of the total prefactor in front of the integral is:

$$\frac{\Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2}-\mathbf{i}P)\Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2}+\mathbf{i}P)}{\Gamma_{\frac{\gamma}{2}}(Q-\mathbf{i}P)\Gamma_{\frac{\gamma}{2}}(Q+\mathbf{i}P)}.$$

□

$$\frac{\sin\pi\chi(\mathbf{i}P+\frac{\alpha}{2})}{\sin\pi\chi\mathbf{i}P}M_\alpha(P-\mathbf{i}\chi,P')+\frac{\sin\pi\chi(\mathbf{i}P-\frac{\alpha}{2})}{\sin\pi\chi\mathbf{i}P}M_\alpha(P+\mathbf{i}\chi,P') = 2\cos(\pi\chi\mathbf{i}P')M_\alpha(P,P')$$

at $\chi = \frac{\gamma}{2}$ and $P = \frac{2\mathbf{i}n}{\gamma}$ we can derive $M_\alpha(P_{-1,n},P') = M_\alpha(P_{1,n},P')$.

Actually it may be easier to just write the shift equation directly on the modular kernel. We can write:

$$\mathcal{M}_\alpha(P,P') = C(\gamma,P',\alpha)\frac{\Gamma_{\frac{\gamma}{2}}(Q-\mathbf{i}P)\Gamma_{\frac{\gamma}{2}}(Q+\mathbf{i}P)}{\Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2}-\mathbf{i}P)\Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2}+\mathbf{i}P)}M_\alpha(P,P')$$

Using the notation $g(P)=\frac{1}{\Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2}-\mathbf{i}P)\Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2}+\mathbf{i}P)}$ and the shift relation of the double gamma function we can obtain:

$$g(P+\mathbf{i}\chi)=\chi^{2\mathbf{i}\chi P-\chi^2}\frac{\Gamma(\chi(Q-\frac{\alpha}{2}-\mathbf{i}P))}{\Gamma(\chi(Q-\frac{\alpha}{2}+\mathbf{i}P-\chi))}g(P),$$
$$g(P-\mathbf{i}\chi)=\chi^{-2\mathbf{i}\chi P-\chi^2}\frac{\Gamma(\chi(Q-\frac{\alpha}{2}+\mathbf{i}P))}{\Gamma(\chi(Q-\frac{\alpha}{2}-\mathbf{i}P-\chi))}g(P).$$

Appendix A. Identities on special functions

A.1. **Gamma and double gamma functions.** The gamma function is $\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt$ for $\Re z>0$. In particular, $\Gamma(n)=(n-1)!$ for $n\in\mathbb{N}$. It has meromorphic extension to $\mathbb{C}$ with simple poles at $\{0,-1,-2,\cdots\}$ and satisfies $\Gamma(z+1)=z\Gamma(z)$, the reflection formula

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin(\pi z)}\qquad\text{for } z\notin\mathbb{Z}, \tag{A.1}$$

For $\mathrm{Re}(z)>0$ the double gamma function $\Gamma_{\frac{\gamma}{2}}(z)$ is:

$$\log\Gamma_{\frac{\gamma}{2}}(z):=\int_0^\infty\frac{dt}{t}\Big[\frac{e^{-zt}-e^{-\frac{Qt}{2}}}{(1-e^{-\frac{\gamma t}{2}})(1-e^{-\frac{2t}{\gamma}})}-\frac{(\frac{Q}{2}-z)^2}{2}e^{-t}+\frac{z-\frac{Q}{2}}{t}\Big]. \tag{A.2}$$

The function $\Gamma_{\frac{\gamma}{2}}(z)$ admits meromorphic extension to $\mathbb{C}$ which has no zeros and has simple poles at $\{-\frac{\gamma n}{2}-\frac{2m}{\gamma}\mid n,m\in\mathbb{N}\}$. Moreover

$$\Gamma_{\frac{\gamma}{2}}(z+\chi)=\sqrt{2\pi}\chi^{\chi z-\frac{1}{2}}\Gamma(\chi z)^{-1}\Gamma_{\frac{\gamma}{2}}(z)\quad\text{for } \chi\in\{\frac{\gamma}{2},\frac{2}{\gamma}\}, \tag{A.3}$$

which implies

$$\Gamma_{\frac{\gamma}{2}}(z+Q)=2\pi\frac{\chi^{(\chi-\chi^{-1})z}}{\Gamma(\chi z)\Gamma(\chi^{-1}z)z}\Gamma_{\frac{\gamma}{2}}(z). \tag{A.4}$$

For $\gamma^2\notin\mathbb{Q}$, $\Gamma_{\frac{\gamma}{2}}(z)$ is completely specified by (A.3) and $\Gamma_{\frac{\gamma}{2}}(\frac{Q}{2})=1$. Other values of γ can be recovered by continuity. We also use the function

$$S_{\frac{\gamma}{2}}(z):=\Gamma_{\frac{\gamma}{2}}(z)\Gamma_{\frac{\gamma}{2}}(Q-z)^{-1}. \tag{A.5}$$

and its functional equation:

$$\frac{S_{\frac{\gamma}{2}}(x+\chi)}{S_{\frac{\gamma}{2}}(x)}=2\sin(\pi\chi x),\quad\text{for } \chi\in\{\frac{\gamma}{2},\frac{2}{\gamma}\}. \tag{A.6}$$

The now give a lemma giving the asymptotic of $S_{\frac{\gamma}{2}}$.

Lemma A.1. We have that

$$\lim_{\Im(x)\to\infty}e^{\mathbf{i}\frac{\pi}{2}x(x-Q)}S_{\frac{\gamma}{2}}(x)=1,$$
$$\lim_{\Im(x)\to-\infty}e^{-\mathbf{i}\frac{\pi}{2}x(x-Q)}S_{\frac{\gamma}{2}}(x)=1.$$

As a consequence it is easy to derive the following bound. Fix $a\in\mathbb{R}$ away from the poles of $S_{\frac{\gamma}{2}}$. Then one has:

$$|S_{\frac{\gamma}{2}}(a+\mathbf{i}t)|\le C_a e^{\frac{\pi}{2}|t(Q-2a)|},\quad\forall t\in\mathbb{R}.$$

A.2. **Convergence of contour integrals.** Throughout this paper we consider in many places expressions of the form

$$\int_{\mathcal{C}} dr e^{\mathbf{i}\pi r\xi} \prod_{i=1}^{N} \frac{S_{\frac{\gamma}{2}}(u_i + r)}{S_{\frac{\gamma}{2}}(v_i + r)},$$

where the u_i, v_i, ξ can a priori be arbitrary parameters in $\mathbb{C}$. Here we discuss how the contour $\mathcal{C}$ is chosen and what are the conditions for the convergence of such integrals. Using the properties of the $S_{\frac{\gamma}{2}}$ function, we see that the integrand has a pole when:

$$r = -u_i - \frac{m\gamma}{2} - \frac{2n}{\gamma}, \quad r = Q - v_i + \frac{m\gamma}{2} + \frac{2n}{\gamma}, \quad m, n \geq 0.$$

There is thus N lattices of poles extending to the left (in the positive ∞ direction) and N lattices of poles extending to the right (in the $-\infty$ direction). The contour $\mathcal{C}$ is then chosen to go from $-\mathbf{i}\infty$ to $+\mathbf{i}\infty$, passing to the left of the poles of the lattices extending to the right and vice versa. This will always be possible except in the special case where there is a pole from one of the left lattices collapsing with a pole from the right lattices, in which case the whole function has a pole. Let us now use the asymptotic (A.1) of $S_{\frac{\gamma}{2}}$ to study the asymptotics of these integrals. Write $r = \mathbf{i}t$. First as $t \to +\infty$:

$$\left| e^{\mathbf{i}\pi r\xi} \prod_{i=1}^{N} \frac{S_{\frac{\gamma}{2}}(u_i + r)}{S_{\frac{\gamma}{2}}(v_i + r)} \right| \sim e^{-\pi t \mathrm{Re}(\xi)} \left| \prod_{i=1}^{N} \frac{S_{\frac{\gamma}{2}}(u_i + r)}{S_{\frac{\gamma}{2}}(v_i + r)} \right| = e^{-\pi t \mathrm{Re}(\xi)} \left| \prod_{i=1}^{N} e^{\frac{\mathbf{i}\pi}{2}\left(v_i^2 + (2\mathbf{i}t - Q)v_i - u_i^2 + (Q - 2\mathbf{i}t)u_i\right)} \right|$$

We then get as $t \to +\infty$:

$$\left| e^{\mathbf{i}\pi r\xi} \prod_{i=1}^{N} \frac{S_{\frac{\gamma}{2}}(u_i + r)}{S_{\frac{\gamma}{2}}(v_i + r)} \right| \sim e^{-\pi t \mathrm{Re}(\xi)} \prod_{i=1}^{N} e^{\pi t \mathrm{Re}(u_i - v_i)} e^{\frac{\pi}{2}\mathrm{Im}(u_i^2 - v_i^2)}.$$

The condition for convergence of the integral are thus:

$$\sum_i \mathrm{Re}(u_i - v_i) - \mathrm{Re}(\xi) < 0, \quad \sum_i \mathrm{Re}(v_i - u_i) - \mathrm{Re}(\xi) > 0.$$

We will also need the following lemma.

Lemma A.2. For fixed parameters ξ, a, b, c consider the function:

$$\int_{\mathcal{C}} \frac{dr}{\mathbf{i}} e^{\mathbf{i}\pi r\xi} \frac{S_{\frac{\gamma}{2}}(a + r) S_{\frac{\gamma}{2}}(b + r)}{S_{\frac{\gamma}{2}}(c + r) S_{\frac{\gamma}{2}}(Q + r)}.$$

The parameters need to obey the constraint above namely:

$$\mathrm{Re}(a + b - c) - Q - \mathrm{Re}(\xi) < 0, \quad Q + \mathrm{Re}(c - a - b) - \mathrm{Re}(\xi) > 0.$$

Assuming the parameter are such that this condition holds one then has the following asymptotics, first as $\mathrm{Im}(\xi) \to +\infty$

$$\frac{S_{\frac{\gamma}{2}}(a) S_{\frac{\gamma}{2}}(b)}{S_{\frac{\gamma}{2}}(c)} (1 + O(e^{\frac{\pi \mathbf{i}\gamma\xi}{2}})),$$

and as $\mathrm{Im}(\xi) \to -\infty$:

$$e^{-\pi \mathbf{i} b\xi} \frac{S_{\frac{\gamma}{2}}(b) S_{\frac{\gamma}{2}}(b - a)}{S_{\frac{\gamma}{2}}(c - b)} (1 + O(e^{-\frac{\pi \mathbf{i}\gamma\xi}{2}})) + e^{-\pi \mathbf{i} a\xi} \frac{S_{\frac{\gamma}{2}}(a) S_{\frac{\gamma}{2}}(a - b)}{S_{\frac{\gamma}{2}}(c - a)} (1 + O(e^{-\frac{\pi \mathbf{i}\gamma\xi}{2}})).$$

Note that when $\mathrm{Re}(a) > \mathrm{Re}(b)$ then the term with $e^{-\pi \mathbf{i} b\xi}$ is the leading term and for $\mathrm{Re}(a) < \mathrm{Re}(b)$ it is the term with $e^{-\pi \mathbf{i} a\xi}$.

In order to perform the identification of the following section, we will try to match the poles of both sides. Therefore we need to determine the poles of the meromorphic function given by:

$$I_\alpha(P, P') := \int_{\mathcal{C}} d\xi \frac{S_{\frac{\gamma}{2}}(\frac{\mathbf{i}P'}{2} + \frac{\alpha}{4} + \xi) S_{\frac{\gamma}{2}}(\frac{\mathbf{i}P'}{2} + \frac{\alpha}{4} - \xi)}{S_{\frac{\gamma}{2}}(Q + \frac{\mathbf{i}P'}{2} - \frac{\alpha}{4} + \xi) S_{\frac{\gamma}{2}}(Q + \frac{\mathbf{i}P'}{2} - \frac{\alpha}{4} - \xi)} e^{2\pi P\xi},$$

For this purpose we must now specify how the contour is chosen. The contour $\mathcal{C}$ of the integral goes from $-i\infty$ to $i\infty$ passing to the right of the poles at $r = -\frac{\mathbf{i}P'}{2} - \frac{\alpha}{4} - n\frac{\gamma}{2} - m\frac{2}{\gamma}$, $r = \frac{\mathbf{i}P'}{2} - \frac{\alpha}{4} - n\frac{\gamma}{2} - m\frac{2}{\gamma}$ and to the left of the poles at $r = \frac{\mathbf{i}P'}{2} + \frac{\alpha}{4} + n\frac{\gamma}{2} + m\frac{2}{\gamma}$, $r = -\frac{\mathbf{i}P'}{2} + \frac{\alpha}{4} + n\frac{\gamma}{2} + m\frac{2}{\gamma}$, with $m, n \in \mathbb{N}^2$. Let us check the convergence of the integral at $\pm i\infty$. For this, by Lemma A.1 we find that as $\xi \to +i\infty$ the integrand of the above integral is equivalent to $c_1 e^{2\pi P\xi} e^{i\pi\xi(2Q-\alpha)}$ for some $c_1 \in \mathbb{C}$ independent of ξ. Therefore this imposes the constraint $2Q - \mathrm{Re}(\alpha) + 2\mathrm{Im}(P) > 0$. Similarly for $\xi \to -i\infty$, the integrand is equivalent to $c_2 e^{2\pi P\xi} e^{-i\pi\xi(2Q-\alpha)}$ for some $c_2 \in \mathbb{C}$ independent of ξ. This gives the constraint $2Q - \mathrm{Re}(\alpha) - 2\mathrm{Im}(P) > 0$. These constraints can be summarized by the conditions

$$Q > \frac{\mathrm{Re}(\alpha)}{2}, \qquad \mathrm{Im}(P) \in \left[-Q + \frac{\mathrm{Re}(\alpha)}{2}, Q - \frac{\mathrm{Re}(\alpha)}{2}\right].$$

If we wish to extend the exact formula for the modular kernel to a meromorphic function of all of its parameters to the whole complex plane we need to use the shift equations.

A.3. **Asymptotics on special functions.** Our estimates will be based on the following asymptotic:

Lemma A.3. Fix $\gamma \in (0,2)$, $a, b \in \mathbb{C}$ with $a \neq -\frac{\gamma n}{2} - \frac{2m}{\gamma}$ for any integers $m, n \geq 0$. Then for $P \in \mathbb{R}$ one has

$$\left|\frac{\Gamma_{\frac{\gamma}{2}}(a + \mathbf{i}P)}{\Gamma_{\frac{\gamma}{2}}(b + \mathbf{i}P)}\right| < C e^{c|P|\log|P|}$$

where $C, c > 0$ depend on γ, a, b but not on P.

Proof. We will prove this bound by directly using the integral formula for the double gamma function which we recall now.

$$\Gamma_{\frac{\gamma}{2}}(x) := \exp\left(\int_0^\infty \frac{dt}{t}\left[\frac{e^{-xt} - e^{-\frac{Qt}{2}}}{(1 - e^{-\frac{\gamma t}{2}})(1 - e^{-\frac{2t}{\gamma}})} - \frac{(\frac{Q}{2} - x)^2}{2} e^{-t} + \frac{x - \frac{Q}{2}}{t}\right]\right). \tag{A.7}$$

This formula is valid as long as $\mathrm{Re}(x) > 0$. Without loss of generality we can assume a, b are real. By the using the shift equations of the double gamma function, we can also assume that $0 < a < b$. From here we can write that:

$$\frac{\Gamma_{\frac{\gamma}{2}}(a + \mathbf{i}P)}{\Gamma_{\frac{\gamma}{2}}(b + \mathbf{i}P)} = \exp\left(\int_0^\infty \frac{dt}{t}\left[\frac{(e^{-at} - e^{-bt})e^{-\mathbf{i}Pt}}{(1 - e^{-\frac{\gamma t}{2}})(1 - e^{-\frac{2t}{\gamma}})} + \frac{(a-b)(Q - 2\mathbf{i}P - a - b)}{2} e^{-t} + \frac{a-b}{t}\right]\right).$$

Since $0 < a < b$, for large t the integral is converging and should at most contribute as $c|P|$. Let us now look at how the integrand behaves for small t. We first write:

$$\begin{aligned}
&\frac{(e^{-at} - e^{-bt})e^{-\mathbf{i}Pt}}{(1 - e^{-\frac{\gamma t}{2}})(1 - e^{-\frac{2t}{\gamma}})} + \frac{(a-b)(Q - 2\mathbf{i}P - a - b)}{2} e^{-t} + \frac{a-b}{t} \\
&= \frac{1}{t}(b - a + (\frac{a^2}{2} - \frac{b^2}{2})t + o(t))(1 - \mathbf{i}Pt + o(t))(1 + \frac{\gamma}{4}t + o(t))(1 + \frac{1}{\gamma}t + o(t)) \\
&+ \frac{(a-b)(Q - 2\mathbf{i}P - a - b)}{2}(1 - t + o(t)) + \frac{a-b}{t} = o(1).
\end{aligned}$$

At the next order we will see the integrand is converging at $t = 0$. We only have to worry about the P^2 term. In the above expansion it should appear as $-\frac{1}{t}(b-a)\frac{P^2}{2}t^2$ and give a total contribution of the form:

$$\exp\left(\int_0^{t_0} \frac{dt}{t}\frac{1}{t}(a-b)\frac{P^2}{2}t^2\right) = e^{\frac{(a-b)t_0 P^2}{2}}.$$

By assuming $b > a$ which is possible using shift equations, this argument seems to conclude the proof. ☐

Appendix B. Identities on hyperbolic hypergeometric functions

This appendix summarizes some integral identities on hyperbolic hypergeometric functions used in [PT01] and [Tes16] and originally coming from the thesis [vdB07].

B.1. **Hyperbolic beta integral.** The first identity we will use is the following hyperbolic beta integral.

Lemma B.1 ([PT01, Lemma 15])**.** Consider $\alpha, \beta \in \mathbb{C}$ satisfying the conditions $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$ and $\mathrm{Re}(\alpha+\beta) < Q$. We have that

$$\int_{-\mathbf{i}\infty}^{\mathbf{i}\infty} \frac{d\tau}{i} d\tau e^{2\pi i\tau\beta} \frac{e^{\frac{\pi i}{2}(\tau+\alpha)(\tau+\alpha-Q)}}{e^{\frac{\pi i}{2}\tau(\tau+Q)}} \frac{S_{\frac{\gamma}{2}}(\tau+\alpha)}{S_{\frac{\gamma}{2}}(\tau+Q)} = \frac{e^{\frac{\pi i}{2}\alpha(\alpha-Q)} e^{\frac{\pi i}{2}\beta(\beta-Q)}}{e^{\frac{\pi i}{2}(\alpha+\beta)(\alpha+\beta-Q)}} \frac{S_{\frac{\gamma}{2}}(\alpha) S_{\frac{\gamma}{2}}(\beta)}{S_{\frac{\gamma}{2}}(\alpha+\beta)}$$

which implies

$$\begin{aligned}
&\frac{1}{i}\int_{i\mathbb{R}} d\tau e^{2\pi i\tau\beta} \frac{e^{\frac{i\pi}{2}(\tau+\alpha)(\tau+\alpha-Q)}}{e^{\frac{i\pi}{2}(\tau+Q)\tau}} \frac{S_{\frac{\gamma}{2}}(\tau+\alpha)}{S_{\frac{\gamma}{2}}(\tau+Q)} = \frac{e^{\frac{i\pi}{2}(\alpha^2-\alpha Q+\beta^2-\beta Q)}}{e^{\frac{i\pi}{2}(\alpha+\beta)(\alpha+\beta-Q)}} \frac{S_{\frac{\gamma}{2}}(\alpha) S_{\frac{\gamma}{2}}(\beta)}{S_{\frac{\gamma}{2}}(\alpha+\beta)} \\
&\Rightarrow \int_{i\mathbb{R}} d\tau e^{2\pi i\tau\beta} e^{i\pi\tau(\alpha-Q)} \frac{S_{\frac{\gamma}{2}}(\tau+\alpha)}{S_{\frac{\gamma}{2}}(\tau+Q)} = i e^{\frac{i\pi}{2}\alpha(Q-\alpha)} e^{-i\pi\alpha\beta} \frac{S_{\frac{\gamma}{2}}(\alpha) S_{\frac{\gamma}{2}}(\beta)}{S_{\frac{\gamma}{2}}(\alpha+\beta)}.
\end{aligned}$$

The contour in τ is the imaginary axis, shifted slightly to the half space of negative real part to avoid the pole at $\tau = 0$.

Lets look at the behavior as $\tau \to \pm\mathbf{i}\infty$ of the above integral over τ. Using the asymptotic of the double sine function we obtain:

$$e^{2\pi i\tau\beta} e^{i\pi\tau(\alpha-Q)} \frac{S_{\frac{\gamma}{2}}(\tau+\alpha)}{S_{\frac{\gamma}{2}}(\tau+Q)} \sim \begin{cases} e^{\frac{\mathbf{i}\pi}{2}\alpha(Q-\alpha)} e^{2\pi i\tau\beta} & \text{as} \quad \mathrm{Im}(\tau) \to \infty, \\ e^{\frac{\mathbf{i}\pi}{2}\alpha(\alpha-Q)} e^{2i\pi\tau(\alpha-Q)} e^{2\pi i\tau\beta} & \text{as} \quad \mathrm{Im}(\tau) \to -\infty. \end{cases} \tag{B.1}$$

By using these asymptotics, the integral converges at $\tau \to +\mathbf{i}\infty$ if and only if $\mathrm{Re}(\beta) > 0$. The integral converges at $\tau \to -\mathbf{i}\infty$ if and only if $\mathrm{Re}(\alpha+\beta) < Q$.

Notice also the integrand has poles when τ equals any of the following:

$$\tau = -\alpha - n\frac{\gamma}{2} - m\frac{2}{\gamma}, \quad \tau = n\frac{\gamma}{2} + m\frac{2}{\gamma}.$$

For $\mathrm{Re}(\alpha) > 0$, the contour can be chosen as the imaginary axis avoided the pole at the origin. For more general α the contour would be more complicated..

B.2. **Hyperbolic special functions.** The next class of identities we will use are the transformations of hyperbolic hypergeometric integrals in Lemmas B.6 and B.5, which were derived in [vdB07] by degenerating a symmetry of the general hyperbolic hypergeometric integral under the action of the E_7 Weyl group. For the reader's convenience, we now give a brief primer on hyperbolic hypergeometric functions to make explicit how to extract them in this form from the original source.

The hyperbolic gamma function is defined for $\omega_1, \omega_2 \in \mathbb{C}$ with positive real parts by

$$G(z; \omega_1, \omega_2) = \exp\left(\mathbf{i}\int_0^\infty \Big(\frac{\sin(2zt)}{2\sinh(\omega_1 t)\sinh(\omega_2 t)} - \frac{z}{\omega_1\omega_2 t}\Big)\frac{dt}{t}\right). \tag{B.2}$$

Recalling that the double sine function is defined for $0 < \Re(z) < Q$ by

$$S_{\frac{\gamma}{2}}(z) = \exp\left(\int_0^\infty \Big(\frac{\sinh((Q/2-z)t)}{2\sinh(\frac{\gamma}{4}t)\sinh(\frac{1}{\gamma}t)} - \frac{Q-2z}{t}\Big)\frac{dt}{t}\right),$$

we find that

$$\begin{aligned}
G(\mathbf{i}(Q/2-z); \frac{\gamma}{2}, \frac{2}{\gamma}) &= \exp\left(\mathbf{i}\int_0^\infty \Big(\frac{\sin(\mathbf{i}(Q-2z)t)}{2\sinh(\frac{\gamma}{2}t)\sinh(\frac{2}{\gamma}t)} - \frac{\mathbf{i}(Q/2-z)}{t}\Big)\frac{dt}{t}\right) \\
&= \exp\left(-\int_0^\infty \Big(\frac{\sinh((Q-2z)t)}{2\sinh(\frac{\gamma}{2}t)\sinh(\frac{2}{\gamma}t)} - \frac{Q/2-z}{t}\Big)\frac{dt}{t}\right) \\
&= \exp\left(-\int_0^\infty \Big(\frac{\sinh((Q/2-z)t)}{2\sinh(\frac{\gamma}{4}t)\sinh(\frac{1}{\gamma}t)} - \frac{Q-2z}{t}\Big)\frac{dt}{t}\right) \\
&= S_{\gamma/2}(z)^{-1},
\end{aligned}$$

where we use that $\sinh(x) = -\mathbf{i}\sin(\mathbf{i}x)$ in the first equality and we make the change of variables $t/2 \mapsto t$ in the second equality. This implies that

$$G(z;\gamma/2,2/\gamma) = S_{\frac{\gamma}{2}}(\frac{Q}{2}+\mathbf{i}z)^{-1}.$$

For any $u \in \mathbb{C}^8$, define the integrand

$$I_h(u,z) := \frac{G(\mathbf{i}\frac{Q}{2} \pm 2z;\gamma/2,2/\gamma)}{\prod_{j=1}^8 G(u_j \pm z;\gamma/2,2/\gamma)} = \frac{\prod_{j=1}^8 S_{\frac{\gamma}{2}}(\frac{Q}{2}+\mathbf{i}u_j \pm \mathbf{i}z)}{S_{\frac{\gamma}{2}}(\pm 2\mathbf{i}z)}.$$

The univariate hyperbolic hypergeometric function from [vdB07, Section 4.4.2] is defined for generic parameters u lying in

$$\mathcal{D}_u := \{u \mid u_1 + \cdots + u_8 = 2\mathbf{i}Q\}, \tag{B.3}$$

and $\Im(u_j - \frac{Q}{2}\mathbf{i}) < 0$ by

$$S_h(u) := \int_{\mathbb{R}} I_h(u,z)dz \tag{B.4}$$

and admits meromorphic extension to all u satisfying (B.3). This function obeys the following transformation symmetry under the action of the E_7 Weyl group $W(E_7)$ on $\mathcal{D}_u$. The identities of interest in this work will be certain limits of this transformation symmetry.

Proposition B.2 ([vdB07, Theorem 4.4.1])**.** The univariate hyperbolic hypergeometric function is invariant under permutations of u and satisfies the symmetry

$$S_h(u) = S_h(w\cdot u)\prod_{1\le j<k\le 4} S_{\frac{\gamma}{2}}(-\mathbf{i}u_j - \mathbf{i}u_k)^{-1}\prod_{5\le j<k\le 8} S_{\frac{\gamma}{2}}(-\mathbf{i}u_j - \mathbf{i}u_k)^{-1},$$

where w is the reflection about the hyperplane normal to $(1,1,1,1,-1,-1,-1,-1)$.

The core objects we will use are two limiting versions of the univariate hyperbolic hypergeometric function. For $u \in \mathcal{D}_u$ satisfying $\Im(u_j) < \frac{Q}{2}$, define the hyperbolic Barnes integral by

$$B_h(u) := 2\int_{\mathbb{R}} \frac{\prod_{j=1,2,7,8} S_{\frac{\gamma}{2}}(Q/2+\mathbf{i}u_j+\mathbf{i}z)}{\prod_{j=3,4,5,6} S_{\frac{\gamma}{2}}(Q/2-\mathbf{i}u_j+\mathbf{i}z)}dz. \tag{B.5}$$

Similarly, for $v \in \mathbb{C}^6$ satisfying $\Im(\sum_{i=1}^6 v_i) > Q$, define the hyperbolic Euler integral by

$$E_h(v) := \int_{\mathbb{R}} \frac{\prod_{j=1}^6 S_{\frac{\gamma}{2}}(Q/2+\mathbf{i}u_j \pm \mathbf{i}z)}{S_{\frac{\gamma}{2}}(\pm 2\mathbf{i}z)}dz. \tag{B.6}$$

The Barnes integral

Proposition B.3 ([vdB07, Proposition 4.4.7])**.** We have that

$$B_h(u) = B_h(w\cdot u)\prod_{j=1,2}\prod_{k=3,4} S_{\frac{\gamma}{2}}(-\mathbf{i}u_j - \mathbf{i}u_k)^{-1}\prod_{j=5,6}\prod_{k=7,8} S_{\frac{\gamma}{2}}(-\mathbf{i}u_j - \mathbf{i}u_k)^{-1},$$

where w is the reflection about the hyperplane normal to $(1,1,1,1,-1,-1,-1,-1)$.

These two limits are related by the following transformation identity, which is a limiting version of Proposition B.2.

Proposition B.4 ([vdB07, Theorem 4.4.11])**.** For $u \in \mathcal{D}_u$ satisfying $\Im(u_1 + u_6) < Q$, we have

$$B_h(u) = E_h(u_2 - s, u_7 - s, u_8 - s, u_3 + s, u_4 + s, u_5 + s)\prod_{j=3,4,5} S_{\frac{\gamma}{2}}(-\mathbf{i}u_1 - \mathbf{i}u_j)^{-1}\prod_{j=2,7,8} S_{\frac{\gamma}{2}}(-\mathbf{i}u_6 - \mathbf{i}u_j)^{-1}$$

for

$$s = \frac{1}{2}(u_2+u_6+u_7+u_8) - \mathbf{i}\frac{Q}{2}.$$

B.3. **Transformations of hyperbolic hypergeometric functions.** We now derive two limiting consequences of Proposition B.4 which are used in [Tes16]. Although these may be specialized from more general principles in [vdB07, Section 5.6], we derive them explicitly here for the benefit of the reader.

Lemma B.5 ([Tes16, Equation (B.19)], [vdB07, Theorem 5.6.14])**.** For parameters $\mu_1, \mu_2, \mu_3, \nu_1, \nu_2 \in \mathbb{C}$, for

$$\sigma := \frac{Q}{2} - \frac{1}{2}\sum_{i=1}^{3}\mu_i - \frac{1}{2}\nu_2 \qquad \text{and} \qquad \lambda := \sum_{i=1}^{3}\mu_i + \sum_{i=1}^{2}\nu_i - \frac{Q}{2}, \tag{B.7}$$

we have that

$$\begin{aligned}(\text{B.8})\quad & 2\int_{\mathbf{i}\mathbb{R}} dy \prod_{i=1}^{3} S_{\frac{\gamma}{2}}(\mu_i - y)\prod_{i=1}^{2} S_{\frac{\gamma}{2}}(\nu_i + y) e^{\pi \mathbf{i}\lambda y} e^{-\frac{1}{2}\pi \mathbf{i} y^2} \\ &= \prod_{i=1}^{3} S_{\frac{\gamma}{2}}(\mu_i + \nu_2) e^{\frac{1}{2}\pi\mathbf{i}\lambda^2} e^{\frac{1}{8}\pi\mathbf{i}Q^2} e^{-\frac{1}{2}\pi\mathbf{i}Q(\lambda+\nu_1)} \int_{\mathbf{i}\mathbb{R}} dy \frac{\prod_{i=1}^{3} S_{\frac{\gamma}{2}}(\mu_i + \sigma \pm y) S_{\frac{\gamma}{2}}(\nu_1 - \sigma \pm y)}{S_{\frac{\gamma}{2}}(\pm 2y)} e^{-2\pi \mathbf{i} y^2}.\end{aligned}$$

Proof. Set

$$\begin{aligned}u = \Big(&-\mathbf{i}\nu_2 + \mathbf{i}Q/2, -\mathbf{i}\nu_1 + \mathbf{i}Q/2, -\mathbf{i}\mu_1 + \mathbf{i}Q/2, -\mathbf{i}\mu_2 + \mathbf{i}Q/2, -\mathbf{i}\mu_3 + \mathbf{i}Q/2, -\mathbf{i}\mu_4 + \mathbf{i}Q/2, -\mathbf{i}\nu_3 + \mathbf{i}Q/2, -\mathbf{i}\nu_4 + \mathbf{i}Q/2\Big) \\ &+ S(0,0,0,0,0,2,-1,-1)\end{aligned}$$

and take the $S \to \infty$ limit of the identity of Proposition B.4 multiplied by

$$e^{-\mathbf{i}\pi S^2 - \pi(2\mu_4 + \nu_3 + \nu_4 - 2Q) - \mathbf{i}\frac{\pi}{2}(\nu_4^2 + \nu_3^2 - \mu_4^2 + Q(\mu_4 - \nu_3 - \nu_4))}.$$

On the LHS, by Lemma A.1, we obtain

$$2\int_{\mathbb{R}} \prod_{j=1}^{2} S_{\frac{\gamma}{2}}(\nu_j + \mathbf{i}z) \prod_{j=1}^{3} S_{\frac{\gamma}{2}}(\mu_j - \mathbf{i}z) e^{\frac{\pi}{2}(3Q - 2\nu_3 - 2\nu_4 - 2\mu_4)z - \mathbf{i}\frac{\pi}{2}z^2} dz.$$

For u to lie in $\mathcal{D}_u$, we must have that $4\mathbf{i}Q - \mathbf{i}\sum_{i=1}^{4}(\mu_i + \nu_i) = 2\mathbf{i}Q$ and hence

$$\mu_4 + \nu_3 + \nu_4 = 2Q - \sum_{i=1}^{3}\mu_i - \sum_{i=1}^{2}\nu_i = \frac{3Q}{2} - \lambda.$$

Substituting and changing variables to $y = \mathbf{i}z$, we find

$$-2\mathbf{i}\int_{\mathbf{i}\mathbb{R}} \prod_{j=1}^{2} S_{\frac{\gamma}{2}}(\nu_j + y) \prod_{j=1}^{3} S_{\frac{\gamma}{2}}(\mu_j - y) e^{-\mathbf{i}\pi\lambda y + \mathbf{i}\frac{\pi}{2}y^2} dy.$$

On the RHS, note that $s = -\mathbf{i}\sigma$. Thus, by Lemma A.1, we obtain

$$\prod_{i=1}^{3} S_{\frac{\gamma}{2}}(\nu_2 + \mu_i) e^{-\frac{1}{2}\pi\mathbf{i}\lambda^2} e^{-\frac{1}{8}\pi\mathbf{i}Q^2} e^{\frac{1}{2}\pi\mathbf{i}Q(\lambda+\nu_1)} \int_{\mathbb{R}} \frac{\prod_{i=1}^{3} S_{\frac{\gamma}{2}}(\mu_i + \sigma \pm \mathbf{i}z) S_{\frac{\gamma}{2}}(\nu_1 - \sigma \pm \mathbf{i}z)}{S_{\frac{\gamma}{2}}(\pm 2\mathbf{i}z)} e^{-2\mathbf{i}\pi z^2} dz,$$

which after a change of variables to $y = \mathbf{i}z$ becomes

$$-\mathbf{i}\prod_{i=1}^{3} S_{\frac{\gamma}{2}}(\nu_2 + \mu_i) e^{-\frac{1}{2}\pi\mathbf{i}\lambda^2} e^{-\frac{1}{8}\pi\mathbf{i}Q^2} e^{\frac{1}{2}\pi\mathbf{i}Q(\lambda+\nu_1)} \int_{\mathbf{i}\mathbb{R}} \frac{\prod_{i=1}^{3} S_{\frac{\gamma}{2}}(\mu_i + \sigma \pm y) S_{\frac{\gamma}{2}}(\nu_1 - \sigma \pm y)}{S_{\frac{\gamma}{2}}(\pm 2y)} e^{2\mathbf{i}\pi y^2} dy.$$

Taking the complex conjugate of both sides yields the desired identity. ☐

Lemma B.6 ([Tes16, Equation (D.32)], [vdB07, Theorem 5.6.17])**.** We have that

$$\begin{aligned}(\text{B.9})\quad & 2\int_{\mathbf{i}\mathbb{R}} dz \frac{S_{\frac{\gamma}{2}}(\frac{Q}{4} - \mu + \frac{m}{2} \pm z)}{S_{\frac{\gamma}{2}}(\frac{3Q}{4} - \mu - \frac{m}{2} \pm z)} e^{4\pi\mathbf{i}\xi z} \\ &= e^{2\pi\mathbf{i}(\xi^2 - (\frac{Q}{4} + \frac{m}{2})^2 + \mu^2)} S_{\frac{\gamma}{2}}(\frac{Q}{2} - m \pm 2\xi) \int_{\mathbf{i}\mathbb{R}} dy \frac{S_{\frac{\gamma}{2}}(\frac{Q}{4} + \frac{m}{2} \pm \mu \pm \xi \pm y)}{S_{\frac{\gamma}{2}}(\pm 2y)} e^{-2\pi\mathbf{i}y^2}.\end{aligned}$$

Proof. Choose the parameters

$$u = \Big(\mathbf{i}Q/4 + \mathbf{i}m/2 + \mathbf{i}\xi, \mathbf{i}Q/4 - \mathbf{i}\mu - \mathbf{i}m/2, \mathbf{i}Q/4 + \mathbf{i}\mu - \mathbf{i}m/2, \mathbf{i}Q/4 - \mathbf{i}\mu - \mathbf{i}m/2, \\ \mathbf{i}Q/4 + \mathbf{i}m/2 + \mathbf{i}\xi, \mathbf{i}Q/4 + \mathbf{i}m/2 - \mathbf{i}\xi, \mathbf{i}Q/4 + \mathbf{i}m/2 - \mathbf{i}\xi, \mathbf{i}Q/4 + \mathbf{i}\mu - \mathbf{i}m/2\Big) \\ + S(-1,0,0,0,1,-1,1,0)$$

and take the $S \to \infty$ limit of the identity of Proposition B.4 multiplied by $e^{\pi(2m+Q)S}$. On the LHS, by Lemma A.1, we obtain

$$2\int_{\mathbb{R}} S_{\frac{\gamma}{2}}(Q/4 \pm \mu + m/2 \pm \mathbf{i}z)e^{-4\pi\xi z}dz = -2\mathbf{i}\int_{\mathbf{i}\mathbb{R}} S_{\frac{\gamma}{2}}(Q/4 \pm \mu + m/2 \pm y)e^{4\pi\mathbf{i}\xi y}dy.$$

On the RHS, applying Lemma A.1, we obtain

$$e^{2\pi\mathbf{i}(\xi^2 - (\frac{Q}{4}+\frac{m}{2})^2 + \mu^2)} S_{\frac{\gamma}{2}}(Q/2 - m \pm 2\xi)\int_{\mathbb{R}} \frac{S_{\frac{\gamma}{2}}(Q/4 \pm \mu + m/2 \pm \xi \pm \mathbf{i}z)}{S_{\frac{\gamma}{2}}(\pm 2\mathbf{i}z)} e^{2\mathbf{i}\pi z^2}dz,$$

which after the change of variables $y = \mathbf{i}z$ becomes

$$-\mathbf{i}e^{2\pi\mathbf{i}(\xi^2 - (\frac{Q}{4}+\frac{m}{2})^2 + \mu^2)} S_{\frac{\gamma}{2}}(Q/2 - m \pm 2\xi)\int_{\mathbf{i}\mathbb{R}} \frac{S_{\frac{\gamma}{2}}(Q/4 \pm \mu + m/2 \pm \xi \pm y)}{S_{\frac{\gamma}{2}}(\pm 2y)} e^{-2\mathbf{i}\pi y^2}dy.$$

Matching the two expressions completes the proof. □

Lemma B.7. The following identity holds:

$$\int_{i\mathbb{R}} S_{\frac{\gamma}{2}}(\frac{\alpha}{2}+r)S_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2}+r)S_{\frac{\gamma}{2}}(iP-r)S_{\frac{\gamma}{2}}(-iP'-r)S_{\frac{\gamma}{2}}(-r)e^{i\pi r(iP'-iP-\frac{Q}{2})}e^{\frac{i\pi r^2}{2}}dr \\ = \mathcal{C}\int_{i\mathbb{R}} d\xi \frac{S_{\frac{\gamma}{2}}(\frac{\mathbf{i}P'}{2}+\frac{\alpha}{4}+\xi)S_{\frac{\gamma}{2}}(\frac{\mathbf{i}P'}{2}+\frac{\alpha}{4}-\xi)}{S_{\frac{\gamma}{2}}(Q+\frac{\mathbf{i}P'}{2}-\frac{\alpha}{4}+\xi)S_{\frac{\gamma}{2}}(Q+\frac{\mathbf{i}P'}{2}-\frac{\alpha}{4}-\xi)}e^{2\pi P\xi},$$

where $\mathcal{C}$ has expression given by:

$$\mathcal{C} = e^{\frac{\mathbf{i}\pi}{2}(\Delta_\alpha - 2PP')}\frac{S_{\frac{\gamma}{2}}(-iP'+Q-\frac{\alpha}{2})S_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2})}{S_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2}-iP)}.$$

Proof. We first apply the complex conjugate of Lemma B.5 with the parameters

$$\mu_1 = iP, \quad \mu_2 = -iP', \quad \mu_3 = 0, \quad \nu_1 = \frac{\alpha}{2}, \quad \nu_2 = Q - \frac{\alpha}{2}, \quad \lambda = iP - iP' + \frac{Q}{2}.$$

With this choice the balancing condition (B.7) is satisfied for $\sigma = \frac{iP'}{2} - \frac{iP}{2} + \frac{\alpha}{4}$. This thus transforms the LHS of the desired identity into

$$\frac{1}{2}S_{\frac{\gamma}{2}}(iP+Q-\frac{\alpha}{2})S_{\frac{\gamma}{2}}(-iP'+Q-\frac{\alpha}{2})S_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2})e^{-\frac{1}{2}\pi i(iP-iP'+\frac{Q}{2})^2}e^{-\frac{1}{8}\pi iQ^2}e^{\frac{1}{2}\pi iQ(iP-iP'+\frac{Q}{2}+\frac{\alpha}{2})} \\ \times\int_{i\mathbb{R}} dy\frac{S_{\frac{\gamma}{2}}(iP+\sigma\pm y)S_{\frac{\gamma}{2}}(-iP'+\sigma\pm y)S_{\frac{\gamma}{2}}(\sigma\pm y)S_{\frac{\gamma}{2}}(\frac{\alpha}{2}-\sigma\pm y)}{S_{\frac{\gamma}{2}}(\pm 2y)}e^{2\pi iy^2} \\ = \frac{1}{2}S_{\frac{\gamma}{2}}(iP+Q-\frac{\alpha}{2})S_{\frac{\gamma}{2}}(-iP'+Q-\frac{\alpha}{2})S_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2})e^{-\frac{1}{2}\pi i(iP-iP'+\frac{Q}{2})^2}e^{-\frac{1}{8}\pi iQ^2}e^{\frac{1}{2}\pi iQ(iP-iP'+\frac{Q}{2}+\frac{\alpha}{2})} \\ \times\int_{i\mathbb{R}} dy\frac{S_{\frac{\gamma}{2}}(\frac{\alpha}{4}\pm\frac{iP}{2}\pm\frac{iP'}{2}\pm y)}{S_{\frac{\gamma}{2}}(\pm 2y)}e^{2\pi iy^2}$$

We are now going to apply Lemma B.6 with parameters $m = \frac{\alpha}{2} - \frac{Q}{2}$, $\mu = -\frac{iP'}{2}$, $\xi = \frac{iP}{2}$. This gives

$$\int_{i\mathbb{R}} dz\frac{S_{\frac{\gamma}{2}}(\frac{iP'}{2}+\frac{\alpha}{4}\pm z)}{S_{\frac{\gamma}{2}}(Q+\frac{iP'}{2}-\frac{\alpha}{4}\pm z)}e^{2\pi Pz} = \frac{1}{2}e^{\pi i(\frac{P^2}{2}+\frac{\alpha^2}{8}+\frac{P'^2}{2})}S_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2}\pm iP)\int_{i\mathbb{R}} dy\frac{S_{\frac{\gamma}{2}}(\frac{\alpha}{4}\pm\frac{iP}{2}\pm\frac{iP'}{2}\pm y)}{S_{\frac{\gamma}{2}}(\pm 2y)}e^{2\pi iy^2}.$$

Therefore this implies the claim of the lemma for the constant:

$$\mathcal{C} = \frac{S_{\frac{\gamma}{2}}(iP + Q - \frac{\alpha}{2}) S_{\frac{\gamma}{2}}(-iP' + Q - \frac{\alpha}{2}) S_{\frac{\gamma}{2}}(Q - \frac{\alpha}{2})}{S_{\frac{\gamma}{2}}(Q - \frac{\alpha}{2} + iP) S_{\frac{\gamma}{2}}(Q - \frac{\alpha}{2} - iP)} e^{-\pi i(\frac{P^2}{2} + \frac{\alpha^2}{8} + \frac{P'^2}{2})} e^{-\frac{1}{2}\pi i(iP - iP' + \frac{Q}{2})^2} e^{-\frac{1}{8}\pi i Q^2} e^{\frac{1}{2}\pi i Q(iP - iP' + \frac{Q}{2} + \frac{\alpha}{2})}$$

$$= \frac{S_{\frac{\gamma}{2}}(-iP' + Q - \frac{\alpha}{2}) S_{\frac{\gamma}{2}}(Q - \frac{\alpha}{2})}{S_{\frac{\gamma}{2}}(Q - \frac{\alpha}{2} - iP)} e^{-\pi i(\frac{P^2}{2} + \frac{\alpha^2}{8} + \frac{P'^2}{2})} e^{-\frac{1}{2}\pi i(iP - iP' + \frac{Q}{2})^2} e^{-\frac{1}{8}\pi i Q^2} e^{\frac{1}{2}\pi i Q(iP - iP' + \frac{Q}{2} + \frac{\alpha}{2})}. \qquad \square$$

Remark. On the right hand side the integral converges at $\pm \mathbf{i}\infty$ if the parameters obey the condition:

$$Q > \frac{\mathrm{Re}(\alpha)}{2}, \qquad \mathrm{Im}(P) \in \left[-Q + \frac{\mathrm{Re}(\alpha)}{2}, Q - \frac{\mathrm{Re}(\alpha)}{2} \right].$$

Also the contour $\mathcal{C}$ of the integral goes from $-i\infty$ to $i\infty$ passing to the right of the poles at $r = -\frac{\mathbf{i}P'}{2} - \frac{\alpha}{4} - n\frac{\gamma}{2} - m\frac{2}{\gamma}$, $r = \frac{\mathbf{i}P'}{2} - \frac{\alpha}{4} - n\frac{\gamma}{2} - m\frac{2}{\gamma}$ and to the left of the poles at $r = \frac{\mathbf{i}P'}{2} + \frac{\alpha}{4} + n\frac{\gamma}{2} + m\frac{2}{\gamma}$, $r = -\frac{\mathbf{i}P'}{2} + \frac{\alpha}{4} + n\frac{\gamma}{2} + m\frac{2}{\gamma}$, with $m, n \in \mathbb{N}^2$. Let us check the convergence of the integral at $\pm i\infty$.

These conditions should somehow to contained in the identities we are citing to prove this result.

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